**MATH 260 – EXAM #2** **Name:** KEY

**Spring Semester 2019**

**Directions:** ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview/natural view calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

*(7 points) 1.* Use elementary row operations to find det($A$): $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 6 \\ 4 & 1 & 12 \end{bmatrix}$

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 6 \\ 4 & 1 & 12 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} -\begin{vmatrix} -1 & 2 & 6 \\ 2 & 1 & 3 \\ 4 & 1 & 12 \end{vmatrix} \underset{4R_1+R_3 \to R_3}{\overset{2R_1+R_2 \to R_2}{=}} -\begin{vmatrix} -1 & 2 & 6 \\ 0 & 5 & 15 \\ 0 & 9 & 36 \end{vmatrix}$$

$$\overset{\frac{1}{5}R_2}{=} -5\begin{vmatrix} -1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 9 & 36 \end{vmatrix} \overset{-9R_2+R_3 \to R_3}{=} -5\begin{vmatrix} -1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{vmatrix}$$

$$= (-5)((-1)(1)(9)) = 45$$

*(4 points) 2.* Use cofactors to find det($A$): $A = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 3 & 4 \\ 2 & 0 & -3 \end{bmatrix}$

$$2\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} + (-3)\begin{vmatrix} 3 & 2 \\ -2 & 3 \end{vmatrix}$$

$$= 2(8-15) + (-3)(9+4)$$

$$= 2(-7) + (-3)(13) = -14 - 39 = -53$$

3. Given the following matrix $A$.

$$A = \begin{bmatrix} 6 & -1 & 0 \\ 2 & -2 & 1 \\ 3 & 0 & -3 \end{bmatrix}$$

*(6 points)* *a.* Find $\operatorname{adj}(A)$.

$$M_c = \begin{bmatrix} 6 & 9 & 6 \\ -3 & -18 & -3 \\ -1 & -6 & -10 \end{bmatrix}$$

$$\operatorname{adj}(A) = M_c^T = \begin{bmatrix} 6 & -3 & -1 \\ 9 & -18 & -6 \\ 6 & -3 & -10 \end{bmatrix}$$

*(3 points)* *b.* Use $\operatorname{adj}(A)$ to determine $A^{-1}$.

$$\det(A) = 6(6) + 9(-1) = 36 - 9 = 27$$

$$A^{-1} = \frac{1}{27}\begin{bmatrix} 6 & -3 & -1 \\ 9 & -18 & -6 \\ 6 & -3 & -10 \end{bmatrix}$$

*(3 points)* 4. What does it mean for a set of vectors, $S$, to be a basis for a vector space, $V$?

$S$ is linearly independent.

$S$ spans $V$.

*(7 points)* 5. Use Cramer's Rule to solve the following system of linear equations.

$$-2x+4y+\ z=-5$$
$$3x-2y-\ z=2$$
$$4x-3y+2z=1$$

$\left(\frac{-11}{27}, \frac{35}{27}, -\frac{17}{27}\right)$

$$\begin{vmatrix} -2 & 4 & 1 \\ 3 & -2 & -1 \\ 4 & -3 & 2 \end{vmatrix} = -2(-4-3)-4(6+4)+1(-9+8) = -2(-7)-4(10)+1(-1)$$
$$= 14-40-1 = -27$$

$$x = \frac{\begin{vmatrix} -5 & 4 & 1 \\ 2 & -2 & -1 \\ 1 & -3 & 2 \end{vmatrix}}{-27} = \frac{1(-6+2)+1(15-4)+2(10-8)}{-27} = \frac{-4+11+4}{-27} = -\frac{11}{27}$$

$$y = \frac{\begin{vmatrix} -2 & -5 & 1 \\ 3 & 2 & -1 \\ 4 & 1 & 2 \end{vmatrix}}{-27} = \frac{1(3-8)+1(-2+20)+2(-4+15)}{-27} = \frac{-5+18+22}{-27} = -\frac{35}{27}$$

$$z = \frac{\begin{vmatrix} -2 & 4 & -5 \\ 3 & -2 & 2 \\ 4 & -3 & 1 \end{vmatrix}}{-27} = \frac{-2(-2+6)-4(3-8)-5(-9+8)}{-27} = \frac{-8+20+5}{-27} = -\frac{17}{27}$$

*(7 points)* 6. Prove that for any vector $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ and any scalars $c, d \in \mathbb{R}$, $\cancel{(c+d)\vec{u} = c\vec{u} + c\vec{v}}$. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

$$(c+d)\vec{u} = (c+d)(u_1, u_2, u_3) = ((c+d)u_1, (c+d)u_2, (c+d)u_3)$$
$$= (cu_1+du_1, cu_2+du_2, cu_3+du_3)$$
$$= (cu_1, cu_2, cu_3) + (du_1, du_2, du_3)$$
$$= c(u_1, u_2, u_3) + d(u_1, u_2, u_3)$$
$$= c\vec{u} + d\vec{u}$$

*(7 points)* 7. Prove that for any three vectors $\vec{p},\vec{q},\vec{r}\in P_2$, $\vec{p}+(\vec{q}+\vec{r})=(\vec{p}+\vec{q})+\vec{r}$.

$\vec{p}=p_0+p_1x+p_2x^2$, $\vec{q}=q_0+q_1x+q_2x^2$, $\vec{r}=r_0+r_1x+r_2x^2$

$$\begin{aligned}
\vec{p}+(\vec{q}+\vec{r}) &= (p_0+p_1x+p_2x^2)+\big((q_0+q_1x+q_2x^2)+(r_0+r_1x+r_2x^2)\big)\\
&= (p_0+p_1x+p_2x^2)+\big((q_0+r_0)+(q_1+r_1)x+(q_2+r_2)x^2\big)\\
&= \big(p_0+(q_0+r_0)\big)+\big(p_1+(q_1+r_1)\big)x+\big(p_2+(q_2+r_2)\big)x^2\\
&= \big((p_0+q_0)+r_0\big)+\big((p_1+q_1)+r_1\big)x+\big((p_2+q_2)+r_2\big)x^2\\
&= (p_0+q_0)+(p_1+q_1)x+(p_2+q_2)x^2+(r_0+r_1x+r_2x^2)\\
&= \big((p_0+p_1x+p_2x^2)+(q_0+q_1x+q_2x^2)\big)+(r_0+r_1x+r_2x^2)\\
&= (\vec{p}+\vec{q})+\vec{r}
\end{aligned}$$

8. Show why the following sets are not vector spaces.

*(4 points)* a. $V=\{(a,b,c)\mid a,b,c\in\mathbb{R} \text{ and } abc=1\}$, together with the standard operations of vector addition and scalar multiplication in $\mathbb{R}^3$.

$\vec{u}=(1,1,1)$ $\vec{v}=(\frac{1}{2},2,1)$ $\vec{u}+\vec{v}=(\frac{3}{2},3,2)$

But $\frac{3}{2}\cdot 3\cdot 2=9$ so, $\vec{u}+\vec{v}\notin V$.

So, V is not closed under addition

$\therefore$ V is not a vector space.

*(4 points)* b. $V=\left\{\begin{bmatrix} a & b \\ 1 & c\end{bmatrix}\middle| a,b,c\in\mathbb{R}\right\}$, together with the standard operations of vector addition and scalar multiplication in $M_{2,2}$.

$A=\begin{bmatrix} 2 & 3 \\ 1 & 4\end{bmatrix}$ and $5\begin{bmatrix} 2 & 3 \\ 1 & 4\end{bmatrix}=\begin{bmatrix} 10 & 15 \\ 5 & 20\end{bmatrix}\notin V$

So, V is not closed under scalar multiplication

$\therefore$ V is not a vector space.

9. Given that $A$ and $B$ are $3\times 3$ matrices and that $\det(A)=4$ and $\det(B)=7$.

*(1 point)* *a.* Find $\det(3A)$. $3^3 \cdot 4 = 108$

*(1 point)* *b.* Find $\det(AB)$. $4 \cdot 7 = 28$

*(1 point)* *c.* Find $\det(A^T)$. $4$

*(1 point)* *d.* Find $\det(B^{-1})$. $1/7$

*(1 point)* *e.* Find $\det(A^3)$. $4^3 = 64$

*(1 point)* *f.* If matrix $C$ is the result of performing $5R_3 + R_2 \to R_2$ to matrix $A$, find $\det(C)$. $4$

*(1 point)* *g.* If matrix $D$ is the result of performing $3R_3$ to matrix $A$, find $\det(D)$. $3 \cdot 4 = 12$

*(1 point)* *h.* If matrix $E$ is the result of performing $R_2 \leftrightarrow R_3$ to matrix $B$, find $\det(E)$. $-7$

*(7 points)* *10.* Prove that $V = \{A \in M_{2,2} \,|\, A \text{ is an upper triangular matrix}\}$ is a subspace of $M_{2,2}$.

$$A = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \quad B = \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ 0 & a_3+b_3 \end{bmatrix} \in V$$

V is cloud under addition

$$kA = k\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} = \begin{bmatrix} ka_1 & ka_2 \\ 0 & ka_3 \end{bmatrix} \in V$$

V is closed under scalar multplication

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$$

V is nonemty

$\therefore$ V is a subspace of $M_{2,2}$.

*(5 points) 11.* Determine if $\{(1,2,3),(1,-1,2),(1,-4,1)\}$ are linearly independent or linearly dependent in $\mathbb{R}^3$.

$c_1(1,2,3)+c_2(1,-1,2)+c_3(1,-4,1)=(0,0,0)$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -4 \\ 3 & 2 & 1 \end{vmatrix} = 1(-1+8)-1(2+12)+1(4+3)$$
$$= 7-14+7=0$$

$\therefore$ The vectors are linearly dependent.

*(5 points) 12.* Determine if $\{1-x+x^2, 2+5x+3x^2, 4-2x+x^2\}$ spans $P_2$.

$c_1(1-x+x^2)+c_2(2+5x+3x^2)+c_3(4-2x+x^2)=0+0x+0x^3$

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 5 & 3 \\ 4 & -2 & 1 \end{vmatrix} = 1(5+6)+1(2-12)+1(-4-20)$$
$$= 11-10-24=-23 \neq 0$$

$\therefore$ The vectors span $P_2$.

*(10 points) 13.* Given a nonempty set $V$ with $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $V$ and with $c$ and $d$ as scalars, write the ten vector space properties that must be verified to conclude that $V$ is a vector space.

| | |
|---|---|
| $\vec{u}+\vec{v}\in V$ | $c\vec{u}\in V$ |
| $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ | $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ |
| $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ | $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ |
| $\vec{u}+\vec{0}=\vec{u}$ | $(cd)\vec{u}=c(d\vec{u})$ |
| $\vec{u}+(-\vec{u})=\vec{0}$ | $1\vec{u}=\vec{u}$ |

*14.* Explain why the set $S$ is not a basis for the vector space $V$.

*(3 points)* *a.* $V = \mathbb{R}^2;\ S = \{(1,2),(3,6)\}$

They are linearly dependent.
or They are the same vector so They do not span $\mathbb{R}^2$

*(3 points)* *b.* $V = \mathbb{R}^3;\ \{(3,1,-1),(1,5,2),(-1,2,4),(2,4,5)\}$

Too many vectors in $\mathbb{R}^3$ so they are linear dependent

*(3 points)* *c.* $V = M_{2,2};\ S = \left\{ \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 6 \\ 1 & 8 \end{bmatrix} \right\}$

Too few vectors in $M_{2,2}$ (4 are needed to span $M_{2,2}$)

So, $S$ does not span $M_{2,2}$

*(6 points)* *15.* Determine if $S = \{(1,-1,1),(2,5,-2),(3,11,-5)\}$ is a basis for $\mathbb{R}^3$.

$$c_1(1,-1,1) + c_2(2,5,-2) + c_3(3,11,-5) = (0,0,0)$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 11 \\ 1 & -2 & -5 \end{vmatrix} = 1(-25+22) - 2(5-11) + 3(2-5)$$
$$= -3 + 12 - 9 = 0$$

So, $S$ is not linearly independent and does not span $\mathbb{R}^3$

$\therefore$ $S$ is not a basis for $\mathbb{R}^3$.