MATH 260 – EXAM #2 **Name:** Key

Winter Session 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 103 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may not use a calculator on this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(7 points) *1.* Use elementary row operations to find det(A): $A = \begin{bmatrix} 8 & -2 & 5 \\ 5 & -2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$

$$\begin{vmatrix} 8 & -2 & 5 \\ 5 & -2 & 1 \\ -1 & 2 & 3 \end{vmatrix} \overset{R_1 \leftrightarrow R_3}{=} -\begin{vmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ 8 & -2 & 5 \end{vmatrix} \underset{8R_1+R_3 \to R_3}{\overset{5R_1+R_2 \to R_2}{=}} -\begin{vmatrix} -1 & 2 & 3 \\ 0 & 8 & 16 \\ 0 & 14 & 29 \end{vmatrix} \overset{\frac{1}{8}R_2}{=} -8\begin{vmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & 29 \end{vmatrix}$$

$$\overset{-14R_2+R_3}{=} -8\begin{vmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -8(-1 \cdot 1 \cdot 1) = 8$$

(5 points) *2.* Use cofactors to find det(A): $A = \begin{bmatrix} 4 & 1 & -3 \\ 5 & 2 & 0 \\ 3 & 6 & 2 \end{bmatrix}$

3rd column:

$$-3\begin{vmatrix} 5 & 2 \\ 3 & 6 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}$$

$$= -3(30-6) + 2(8-5)$$

$$= -3(24) + 2(3) = -72 + 6 = -66$$

3. Given the following matrix *A*.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & -2 & 1 \\ 0 & 2 & 8 \end{bmatrix}$$

(6 points) *a.* Find $\text{adj}(A)$.

$$M_C = \begin{bmatrix} -18 & -40 & 10 \\ 0 & 24 & -6 \\ 9 & 17 & -11 \end{bmatrix} \qquad \text{adj}(A) = M_C^T = \begin{bmatrix} -18 & 0 & 9 \\ -40 & 24 & 17 \\ 10 & -6 & -11 \end{bmatrix}$$

(3 points) *b.* Use $\text{adj}(A)$ to determine A^{-1}.

$$\det(A) = 3(-18) + 5(0) + 0(9) = -54$$

$$A^{-1} = -\frac{1}{54}\begin{bmatrix} -18 & 0 & 9 \\ -40 & 24 & 17 \\ 10 & -6 & -11 \end{bmatrix}$$

4. Given the following vectors: $\vec{u} = (3,4,1)$, $\vec{v} = (5,3,-2)$

Evaluate the following.

(2 points) *a.* $7\vec{u} - 2\vec{v}$ $= 7(3,4,1) - 2(5,3,-2)$

$= (21, 28, 7) + (-10, -6, 4)$

$= (11, 22, 11)$

(2 points) *b.* $2(4\vec{u} + 3\vec{v})$ $= 8\vec{u} + 6\vec{v} = 8(3,4,1) + 6(5,3,-2)$

$= (24, 32, 8) + (30, 18, -12)$

$= (54, 50, -4)$

(2 points) *c.* Find $\vec{z}$, where $3\vec{u} + 2\vec{z} = 4\vec{v}$

$2\vec{z} = 4\vec{v} - 3\vec{u}$

$\vec{z} = 2\vec{v} - \frac{3}{2}\vec{u}$

$= 2(5,3,-2) - \frac{3}{2}(3,4,1) = (10, 6, -4) + (-\frac{9}{2}, -6, -\frac{3}{2})$

$= (\frac{11}{2}, 0, -\frac{11}{2})$

5. Given that A and B are 4×4 matrices and that $\det(A) = 3$ and $\det(B) = 5$.

(1 point) a. Find $\det(2A)$. $= 2^4 \cdot 3 = 16 \cdot 3 = 48$

(1 point) b. Find $\det(AB)$. $= 3 \cdot 5 = 15$

(1 point) c. Find $\det(B^T)$. $= 5$

(1 point) d. Find $\det(A^{-1})$. $= \frac{1}{3}$

(1 point) e. Find $\det(A^2)$. $= 3^2 = 9$

(1 point) f. If matrix C is the result of performing $3R_1 + R_3 \to R_3$ to matrix B, find $\det(C)$. $= 5$

(1 point) g. If matrix D is the result of performing $5R_4$ to matrix B, find $\det(D)$. $= 5 \cdot 5 = 25$

(1 point) h. If matrix E is the result of performing $R_1 \leftrightarrow R_2$ to matrix A, find $\det(E)$. $= -3$

(7 points) 6. Use Cramer's Rule to solve the following system of linear equations.

$$
\begin{aligned}
4x - 2y + 3z &= -2 \\
2x + 2y + 5z &= 16 \\
8x - 5y - 2z &= 4
\end{aligned}
$$

$$\begin{vmatrix} 4 & -2 & 3 \\ 2 & 2 & 5 \\ 8 & -5 & -2 \end{vmatrix} = 4(-4+25) + 2(-4-40) + 3(-10-16)$$
$$= 4(21) + 2(-44) + 3(-26) = 84 - 88 - 78 = -82$$

$$x = \frac{\begin{vmatrix} -2 & -2 & 3 \\ 16 & 2 & 5 \\ 4 & -5 & -2 \end{vmatrix}}{-82} = \frac{-2(-4+25) + 2(-32-20) + 3(-80-8)}{-82} = \frac{-2(21) + 2(-52) + 3(-88)}{-82}$$
$$\frac{-42 - 104 - 264}{-82} = \frac{410}{82} = 5$$

$$y = \frac{\begin{vmatrix} 4 & -2 & 3 \\ 2 & 16 & 5 \\ 8 & 4 & -2 \end{vmatrix}}{-82} = \frac{4(-32-20) + 2(-4-40) + 3(8-128)}{-82} = \frac{4(-52) + 2(-44) + 3(-120)}{-82}$$
$$= \frac{-208 - 88 - 360}{-82} = -\frac{656}{82} = 8$$

$$z = \frac{\begin{vmatrix} 4 & -2 & -2 \\ 2 & 2 & 16 \\ 8 & -5 & 4 \end{vmatrix}}{-82} = \frac{4(8+80) + 2(8-128) - 2(-10-16)}{-82} = \frac{4(88) + 2(-120) - 2(-26)}{-82}$$
$$= \frac{352 - 240 + 52}{-82} = -\frac{164}{82} = -2$$

$(5, 8, -2)$

(5 points) *7.* Prove that for any three vectors $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$, and $\vec{w}=(w_1,w_2,w_3)$ in $\mathbb{R}^3$, $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$.

$$\begin{aligned}
\vec{u}+(\vec{v}+\vec{w}) &= (u_1,u_2,u_3)+\big((v_1,v_2,v_3)+(w_1,w_2,w_3)\big)\\
&= (u_1,u_2,u_3)+(v_1+w_1,v_2+w_2,v_3+w_3)\\
&= \big(u_1+(v_1+w_1),u_2+(v_2+w_2),u_3+(v_3+w_3)\big)\\
&= \big((u_1+v_1)+w_1,(u_2+v_2)+w_2,(u_3+v_3)+w_3\big)\\
&= (u_1+v_1,u_2+v_2,u_3+v_3)+(w_1,w_2,w_3)\\
&= \big((u_1,u_2,u_3)+(v_1,v_2,v_3)\big)+(w_1,w_2,w_3)\\
&= (\vec{u}+\vec{v})+\vec{w}
\end{aligned}$$

(5 points) *8.* Prove that for any two vectors $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ in $\mathbb{R}^3$ and any scalar *c*, $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$.

$$\begin{aligned}
c(\vec{u}+\vec{v}) &= c\big((u_1,u_2,u_3)+(v_1,v_2,v_3)\big)\\
&= c(u_1+v_1,u_2+v_2,u_3+v_3)\\
&= \big(c(u_1+v_1),c(u_2+v_2),c(u_3+v_3)\big)\\
&= (cu_1+cv_1,cu_2+cv_2,cu_3+cv_3)\\
&= (cu_1,cu_2,cu_3)+(cv_1,cv_2,cv_3)\\
&= c(u_1,u_2,u_3)+c(v_1,v_2,v_3)\\
&= c\vec{u}+c\vec{v}
\end{aligned}$$

(5 points) *9.* Prove that for any two vectors $\vec{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\vec{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ in $M_{2,2}$, $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

$$\vec{A} + \vec{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \vec{B} + \vec{A}$$

10. Show why the following sets are not vector spaces.

(4 points) *a.* $V = \{(a,b,2) \mid a,b \in \mathbb{R}\}$, together with the standard operations of vector addition and scalar multiplication.

$$(a_1, b_1, 2) + (a_2, b_2, 2) = (a_1+b_1, a_2+b_2, 4) \notin V$$

So V is not closed under addition

$\therefore$ V is not a vector space.

(4 points) *b.* $\mathbb{Q}$ = The set of all rational numbers, together with the usual operations of addition and scalar multiplication.

$$\frac{2}{3} \cdot \pi = \frac{2}{3}\pi \notin \mathbb{Q}$$

So, $\mathbb{Q}$ is not closed under scalar multiplication,

$\therefore$ $\mathbb{Q}$ is not a vector space.

(4 points) *c.* V = The set of all polynomials of degree 4, together with the usual operations of polynomial addition and scalar multiplication.

$(x^4+3x^3+5x+2)-(x^4+2x^3+2x+6) = x^3+3x-4 \notin V$

since the polynomial is of degree 3

So, V is not closed under addition.

$\therefore$ V is not a vector space

(7 points) *11.* Prove that W = The set of 2×2 diagonal matrices is a subspace of $M_{2,2}$.

$W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

$\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 0 \\ 0 & b_1+b_2 \end{bmatrix} \in W$ W is closed under addition

$k \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix} \in W$ W is closed under scalar multiplication

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a diagonal matrix so $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$,

So, W is nonempty.

$\therefore$ W is a subspace of $M_{2,2}$.

(7 points) *12.* Show that the set $S = \{(a,b,0,c) | a,b,c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$.

$(a_1, b_1, 0, c_1) + (a_2, b_2, 0, c_2) = (a_1+a_2, b_1+b_2, 0, c_1+c_2) \in S$,

S is closed under addition.

$k(a, b, 0, c) = (ka, kb, 0, kc) \in S$. S is closed under scalar multiplication

$(0,0,0,0) \in S$, So S is nonempty,

$\therefore$ S is a subspace of $\mathbb{R}^4$.

13. Write the vector $\vec{v}$ as a linear combination of the vectors in the set *S*, if possible.

(5 points) a. $\vec{v} = (4,5,6)$; $S = \{(1,2,3),(3,4,5)\}$

$(4,5,6) = c_1(1,2,3) + c_2(3,4,5)$

$c_1 + 3c_2 = 4$
$2c_1 + 4c_2 = 5$
$3c_1 + 5c_2 = 6$

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -2 & -3 \\ 0 & -4 & -6 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 + 3c_2 = 4$
$c_2 = 3/2$

$c_1 + 3(3/2) = 4$
$c_1 + \frac{9}{2} = 4$
$c_1 = -1/2$

$(4,5,6) = -\frac{1}{2}(1,2,3) + \frac{3}{2}(3,4,5)$

(5 points) b. $\vec{v} = (3,0,-6)$; $S = \{(1,-1,2),(2,4,-2),(1,2,-4)\}$

$(3,0,-6) + c_1(1,-1,2) + c_2(2,4,-2) + c_3(1,2,-4)$

$c_1 + 2c_2 + c_3 = 3$
$-c_1 + 4c_2 + 2c_3 = 0$
$2c_1 - 2c_2 - 4c_3 = -6$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -1 & 4 & 2 & 0 \\ 2 & -2 & -4 & -6 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -6 & -6 & -12 \end{array}\right]$$

$$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & -3 & -9 \end{array}\right] \xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$c_1 + 2c_2 + c_3 = 3$
$2c_2 + c_3 = 1$
$c_3 = 3$
$2c_2 + 3 = 1$
$2c_2 = -2$
$c_2 = -1$
$c_1 + 2(-1) + 3 = 3$
$c_1 = 2$

$(3,0,-6) = 2(1,-1,2) - 1(2,4,-2) + 3(1,2,-4)$

(10 points) 14. Given a nonempty set *V* with $\vec{u}$, $\vec{v}$, and $\vec{w}$ in *V* and with *c* and *d* as scalars, write the ten vector space properties that must be verified to conclude that *V* is a vector space.

1. $\vec{u} + \vec{v}$ is in V
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
4. There exists a $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v}$
5. For every $\vec{v} \in V$ there exists a $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0}$
6. $c\vec{u}$ is in V
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$