MATH 260 – EXAM #3 **Name:** KEY

Spring Semester 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview/natural view calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

1. Given $\vec{x} = (-4,5,2,3)$, $\vec{y} = (2,3,-4,-2)$ in $\mathbb{R}^4$. Find:

(3 points) a. $\vec{x} \cdot \vec{y}$ $= -8 + 15 - 8 - 6 = -7$

(3 points) b. $\|\vec{x}\| = \sqrt{16+25+4+9} = \sqrt{54} = 3\sqrt{6}$

(3 points) c. A unit vector in the direction of $\vec{y}$.

$$\|\vec{y}\| = \sqrt{4+9+16+4} = \sqrt{33}$$

$$\hat{y} = \frac{(2,3,-4,-2)}{\sqrt{33}} = \left(\frac{2}{\sqrt{33}}, \frac{3}{\sqrt{33}}, -\frac{4}{\sqrt{33}}, -\frac{2}{\sqrt{33}}\right)$$

(5 points) 2. Use the Wronskian to show that the functions $f_1(x) = e^{5x}$ and $f_2(x) = e^{-2x}$ are linearly independent on the interval $(-\infty, \infty)$. Explain the result.

$$\begin{vmatrix} e^{5x} & e^{-2x} \\ 5e^{5x} & -2e^{-2x} \end{vmatrix} = -2e^{3x} - 5e^{3x} = -7e^{3x} \neq 0$$

Since the Wronskian is never zero on $(-\infty, \infty)$ $f_1(x)$ and $f_2(x)$ are linearly independent

3. Given $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & -4 \\ 1 & 2 & -6 & 10 \end{bmatrix}$.

(7 points) *a.* Find the row space of A.

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & -4 \\ 1 & 2 & -6 & 10 \end{bmatrix} \xrightarrow[-R_1+R_2\to R_2]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -8 \\ 0 & 1 & -5 & 8 \end{bmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-1R_2} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{s(1,1,-1,2)+t(0,1,-5,8) \mid s,t\in\mathbb{R}\}$

(2 points) *b.* Find a basis for the row space of A.

$\{(1,1,-1,2),(0,1,-5,8)\}$

(3 points) *c.* Find the column space of A.

$\{r(1,2,1)+q(1,1,2) \mid r,q\in\mathbb{R}\}$

(2 points) *d.* Find a basis for the column space of A.

$\{(1,2,1),(1,1,2)\}$

(2 points) *e.* Find rank(A).

2

4. Given $A = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 5 & 5 \end{bmatrix}$.

(7 points) *a.* Find the nullspace of *A*.

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & 4 & 0 \\ 1 & -1 & 2 & 3 & 0 \\ 1 & -2 & 5 & 5 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 4 & 0 \\ 1 & -2 & 5 & 5 & 0 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 0 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & -1 & 3 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 0 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 + 2x_3 + 3x_4 = 0$

$x_2 - 3x_3 - 2x_4 = 0$

$x_4 = t$

$x_3 = s$

$x_2 = 3s + 2t$

$x_1 = 3s + 2t - 2s - 3t$

$= s - t$

$\{(s-t, 3s+2t, s, t) \mid s, t \in \mathbb{R}\} = \{s(1,3,1,0) + t(-1,2,0,1) \mid s, t \in \mathbb{R}\}$

(3 points) *b.* Find a basis for the nullspace of *A*.

$\{(1,3,1,0), (-1,2,0,1)\}$

(2 points) *c.* Find nullity(*A*).

2

(6 points) *5.* Given the coordinate matrix of vector $\bar{x}$ relative to nonstandard basis *B* in $\mathbb{R}^2$ to be $[\bar{x}]_B = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$ where $B = \{(2,1),(1,-2)\}$. Find the coordinate vector of $\bar{x}$ relative to the standard basis in $\mathbb{R}^2$.

$-4(2,1) + 7(1,-2)$

$= (-8,-4) + (7,-14)$

$= (-1, -18)$

(6 points) *6.* Determine the component vector of the vector $\vec{v} = (8,-2)$ in $\mathbb{R}^2$ with the standard basis relative to the ordered basis $B = \{(-1,3),(3,2)\}$

$c_1(-1,3) + c_2(3,2) = (8,-2)$

$-c_1 + 3c_2 = 8$

$3c_1 + 2c_2 = -2$

$$\left[\begin{array}{cc|c} -1 & 3 & 8 \\ 3 & 2 & -2 \end{array}\right] \xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{cc|c} -1 & 3 & 8 \\ 0 & 11 & 22 \end{array}\right] \xrightarrow{\frac{1}{11}R_2} \left[\begin{array}{cc|c} -1 & 3 & 8 \\ 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-1R_1} \left[\begin{array}{cc|c} 1 & -3 & -8 \\ 0 & 1 & 2 \end{array}\right] \quad c_1 - 3c_2 = -8 \quad c_2 = 2 \quad c_1 - 6 = -8 \quad c_1 = -2$$

$$[\vec{v}]_B = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(7 points) *7.* Find the transition matrix P from basis B to basis C where B is the ordered basis $\{(-2,4),(5,-3)\}$ in $\mathbb{R}^2$ and C is the ordered basis $\{(3,1),(-1,2)\}$.

$$\left[\begin{array}{cc|cc} 3 & -1 & -2 & 5 \\ 1 & 2 & 4 & -3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 4 & -3 \\ 3 & -1 & -2 & 5 \end{array}\right] \xrightarrow{-3R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 4 & -3 \\ 0 & -7 & -14 & 14 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 4 & -3 \\ 0 & 1 & 2 & -2 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -2 \end{array}\right] \qquad P = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}$$

8. Given $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1,x_2,x_3) = (x_1 - 5x_2 + 3x_3, 4x_1 + 2x_3)$.

(7 points) *a.* Verify that T is a linear transformation.

$T((x_1,x_2,x_3)+(y_1,y_2,y_3)) = T(x_1+y_1, x_2+y_2, x_3+y_3)$

$= ((x_1+y_1) - 5(x_2+y_2) + 3(x_3+y_3), 4(x_1+y_1) + 2(x_3+y_3)) = (x_1+y_1-5x_2-5y_2+3x_3+3y_3, 4x_1+4y_1+2x_3+2y_3)$

$= (x_1-5x_2+3x_3, 4x_1+2x_3) + (y_1-5y_2+3y_3, 4y_1+2y_3) = T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$

$T(k(x_1,x_2,x_3)) = T(kx_1,kx_2,kx_3) = (kx_1 - 5kx_2 + 3kx_3, 4kx_1 + 2kx_3)$

$= (k(x_1-5x_2+3x_3), k(4x_1+2x_3)) = k(x_1-5x_2+3x_3, 4x_1+2x_3)$

$= kT(x_1,x_2,x_3)$ $\therefore$ T is a linear transformation

(3 points) *b.* Determine the matrix of the linear transformation.

$$\begin{bmatrix} 1 & -5 & 3 \\ 4 & 0 & 2 \end{bmatrix}$$

9. Given the mapping $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -3 \end{bmatrix}$

(7 points) a. Determine $\operatorname{Ker}(T)$.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$

$x_2 - 3x_3 = 0$

Let $x_3 = t$

$x_2 = 3t$

$x_1 = -3t$

$\{t(-3, 3, 1) \mid t \in \mathbb{R}\}$

(3 points) b. Determine $\operatorname{Rng}(T)$.

$\{s(2,1,0) + r(1,1,1) \mid s, r \in \mathbb{R}\}$

(2 points) c. Determine $\dim[\operatorname{Ker}(T)]$. 1

(2 points) d. Determine $\dim[\operatorname{Rng}(T)]$. 2

10. Given $A = \begin{bmatrix} 6 & 3 & -4 \\ -5 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$.

(7 points) *a.* Find the eigenvalues of A.

$$\begin{vmatrix} \lambda-6 & -3 & 4 \\ 5 & \lambda+2 & -2 \\ 0 & 0 & \lambda+1 \end{vmatrix} = (\lambda+1)\left[(\lambda-6)(\lambda+2)+15\right]$$

$$= (\lambda+1)\left[\lambda^2-4\lambda-12+15\right]$$

$$= (\lambda+1)(\lambda^2-4\lambda+3)$$

$$= (\lambda+1)(\lambda-3)(\lambda-1) = 0$$

$$\lambda = -1, 3, 1$$

(7 points) *b.* For each eigenvalue, find the corresponding eigenvectors.

$\lambda = -1$

$$\begin{bmatrix} -7 & -3 & 4 \\ 5 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2\to R_2} \begin{bmatrix} -7 & -3 & 4 \\ -2 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} -7 & -3 & 4 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{7R_2+R_1\to R_1} \begin{bmatrix} 0 & 4 & -3 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 - x_3 = 0$, $4x_2 - 3x_3 = 0$ $\quad x_3 = t$, $x_2 = \frac{3}{4}t$ $\quad x_1 = -\frac{3}{4}t + t$, $x_1 = \frac{1}{4}t$ $\quad \left(\frac{1}{4}, \frac{3}{4}, 1\right)$

$\lambda = 3$

$$\begin{bmatrix} -3 & -3 & 4 \\ 5 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_2+R_1\to R_2} \begin{bmatrix} 2 & 2 & 2 \\ 5 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{-5R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -7 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{7}R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{-4R_2+R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0$, $x_3 = 0$ $\quad x_2 = s$, $x_1 = -s$ $\quad (-1, 1, 0)$

$\lambda = 1$

$$\begin{bmatrix} -5 & -3 & 4 \\ 5 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_1+R_2\to R_2} \begin{bmatrix} -5 & -3 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} -5 & -3 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_2+R_3\to R_3} \begin{bmatrix} -5 & -3 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-5x_1 - 3x_2 - 4x_3 = 0$, $x_3 = 0$ $\quad x_2 = r$ $\quad x_1 = -\frac{3}{5}r$ $\quad \left(-\frac{3}{5}, 1, 0\right)$

(3 points) *c.* For each eigenvalue, find the corresponding eigenspaces.

$E_{-1} = \left\{ t\left(\frac{1}{4}, \frac{3}{4}, 1\right) \mid t \in \mathbb{R} \right\}$

$E_3 = \left\{ s(-1, 1, 0) \mid s \in \mathbb{R} \right\}$

$E_1 = \left\{ r\left(-\frac{3}{5}, 1, 0\right) \mid r \in \mathbb{R} \right\}$