MATH 260 – EXAM #3 **Name:** Key
Winter Session 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may not use a calculator on this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(7 points) 1. Determine if $\{(1,-1,2,3),(2,-1,1,-1),(-1,1,1,1)\}$ are linearly independent or linearly dependent in $\mathbb{R}^4$.

$$c_1(1,-1,2,3)+c_2(2,-1,1,-1)+c_3(-1,1,1,1)=(0,0,0,0)$$

$$\begin{aligned} c_1+2c_2-c_3&=0\\ -c_1-c_2+c_3&=0\\ 2c_1+c_2+c_3&=0\\ 3c_1-c_2+c_3&=0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1&2&-1&0\\ -1&-1&1&0\\ 2&1&1&0\\ 3&-1&1&0 \end{array}\right] \xrightarrow[-3R_1+R_4\to R_4]{R_1+R_2\to R_2,\ -2R_1+R_3\to R_3} \left[\begin{array}{cccc} 1&2&-1&0\\ 0&1&0&0\\ 0&-3&3&0\\ 0&-7&4&0 \end{array}\right] \xrightarrow{3R_2+R_3\to R_3,\ 7R_2+R_4\to R_4}$$

$$\left[\begin{array}{ccc|c} 1&2&-1&0\\ 0&1&0&0\\ 0&0&3&0\\ 0&0&4&0 \end{array}\right] \xrightarrow[\frac{1}{4}R_4]{\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1&2&-1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&1&0 \end{array}\right] \xrightarrow{-1R_3+R_4\to R_4} \left[\begin{array}{ccc|c} 1&2&-1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{array}\right]$$

$$\begin{aligned} c_1+2c_2-c_3&=0\\ c_2&=0\\ c_3&=0\\ c_1&=0 \end{aligned}$$

$\therefore$ S is linearly independent

(7 points) 2. Determine if $\{(2,-1,4),(3,-3,5),(1,1,3)\}$ is a basis for $\mathbb{R}^3$.

$$\begin{vmatrix} 2&3&1\\ -1&-3&1\\ 4&5&3 \end{vmatrix} = 2(-9-5)-3(-3-4)+1(-5+12)$$

$$=2(-14)-3(-7)+1(7)$$

$$=-28+21+7=0$$

$\therefore$ S is linearly dependent and does not span $\mathbb{R}^3$.

$\therefore$ S is not a basis for $\mathbb{R}^3$.

(7 points) 3. Determine if the following matrices are linearly independent or linearly dependent in $M_2(\mathbb{R})$.

$$A_1 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$c_1 + 2c_2 + c_3 = 0$$
$$-c_1 + c_2 - c_3 = 0$$
$$2c_1 + 2c_3 = 0$$
$$3c_2 + c_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \begin{matrix} \frac{1}{3}R_2 \\ -\frac{1}{4}R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[-3R_2+R_4 \to R_4]{-1R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_1 + 2c_2 + c_3 = 0$
$c_2 = 0$
$c_3 = 0$
$c_1 = 0$

$\therefore$ The set is linearly independent

(7 points) 4. Determine if $S = \{1+2x, 3+x^2, 1-2x+x^2\}$ is a basis for P^2.

$$c_1(1+2x) + c_2(3+x^2) + c_3(1-2x+x^2)$$

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} = 1(0+2) - 2(3-1) = 2 - 4 = -2 \neq 0$$

$\therefore$ S are linearly independent and span P_2.

$\therefore$ S is a basis for P_2.

5. Explain why the set S is not a basis for the vector space V.

(2 points) a. $V = \mathbb{R}^3$; $S = \{(1,0,1),(0,1,1),(0,0,0)\}$

S contains the zero vector.

(2 points) b. $V = \mathbb{R}^4$; $\{(3,1,-1,4),(1,5,2,0),(-1,2,4,1)\}$

S does not span $\mathbb{R}^4$ because it has 3 vectors and 4 vectors are needed to span $\mathbb{R}^4$.

(2 points) c. $V = \mathbb{R}^3$; $S = \{(1,0,0),(0,1,1),(2,1,1)\}$

$(2,1,1) = 2(1,0,0) + 1(0,1,1)$. So, S is not linearly independent.

(2 points) d. $V = \mathbb{R}^2$; $S = \{(1,2),(2,5),(3,-1)\}$

There are 3 vectors in S for $\mathbb{R}^2$. So, S is linearly dependent.

6. Given $\vec{x} = (3,2,-5)$, $\vec{y} = (6,1,8)$. Find:

(2 points) a. $\vec{x}\cdot\vec{y} = 18+2-40 = -20$

(2 points) b. $\|\vec{x}\| = \sqrt{9+4+25} = \sqrt{38}$

(3 points) c. A unit vector in the direction of $\vec{y}$.

$\|\vec{y}\| = \sqrt{36+1+64} = \sqrt{101}$

$$\frac{\vec{y}}{\|\vec{y}\|} = \frac{(6,1,8)}{\sqrt{101}} = \left(\frac{6}{\sqrt{101}}, \frac{1}{\sqrt{101}}, \frac{8}{\sqrt{101}}\right)$$

(4 points) 7. Use the Wronskian to show that the functions $f_1(x) = e^{2x}$ and $f_2(x) = e^{3x}$ are linearly independent on the interval $(-\infty,\infty)$. Explain the result.

$$\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0$$

Since the Wronskian is not zero,
f_1 and f_2 are linearly independent.

8. Given $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 6 & -3 & 5 \\ 1 & 2 & -1 & -1 \\ 5 & 10 & -5 & 7 \end{bmatrix}$.

(6 points) *a.* Find the row space of A.

$$\xrightarrow[-5R_1+R_4 \to R_4]{-3R_1+R_2 \to R_2,\ -R_1+R_3 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow[-R_2+R_4 \to R_4]{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{ s(1,2,-1,3) + t(0,0,0,1) \mid s,t \in \mathbb{R} \}$$

(1 point) *b.* Find a basis for the row space of A.

$$\{ (1,2,-1,3),\ (0,0,0,1) \}$$

(3 points) *c.* Find the column space of A.

$$\left\{ q\begin{bmatrix} 1 \\ 3 \\ 1 \\ 5 \end{bmatrix} + r\begin{bmatrix} 3 \\ 5 \\ -1 \\ 7 \end{bmatrix} \,\middle|\, q,r \in \mathbb{R} \right\}$$

(1 point) *d.* Find a basis for the column space of A.

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ -1 \\ 7 \end{bmatrix} \right\}$$

(2 points) *e.* Find rank(A).

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9. Given $A = \begin{bmatrix} 2 & 4 & 3 & -6 \\ 1 & 2 & 2 & -5 \\ 3 & 6 & 5 & -11 \end{bmatrix}$.

(6 points) *a.* Find the nullspace of A.

$$\begin{bmatrix} 2 & 4 & 3 & -6 \\ 1 & 2 & 2 & -5 \\ 3 & 6 & 5 & -11 \end{bmatrix} \xrightarrow[-3R_2+R_3 \to R_3]{-2R_2+R_1 \to R_1} \begin{bmatrix} 0 & 0 & -1 & 4 \\ 1 & 2 & 2 & -5 \\ 0 & 0 & -1 & 4 \end{bmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{bmatrix} 0 & 0 & -1 & 4 \\ 1 & 2 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 2 & -5 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_2} \begin{bmatrix} 1 & 2 & 2 & -5 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 + 2x_3 - 5x_4 = 0$
$x_3 - 4x_4 = 0$
$x_4 = t$
$x_3 = 4t$
$x_2 = s$
$x_1 + 2s + 8t - 5t = 0$
$x_1 = -2s - 3t$

$\{ s(-2,1,0,0) + t(-3,0,4,1) \mid s,t \in \mathbb{R} \}$

(1 point) *b.* Find a basis for the nullspace of A.

$\{ (-2,1,0,0), (-3,0,4,1) \}$

(2 points) *b.* Find nullity(A).

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(4 points) *10.* Determine the component vector of the vector $\vec{v} = (-1,2)$ in $\mathbb{R}^2$ relative to the ordered basis $B = \{(2,2),(0,-1)\}$

$c_1(2,2) + c_2(0,-1) = (-1,2)$

$$\left[\begin{array}{cc|c} 2 & 0 & -1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{-R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 1 & -3 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 2 & -1 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -1/2 & 1 \\ 0 & 1 & -3 \end{array}\right]$$

$c_1 - \frac{1}{2}c_2 = 1$
$c_2 = -3$

$c_1 + \frac{3}{2} = 1$
$c_1 = -1/2$

$$[\vec{v}]_B = \begin{bmatrix} -1/2 \\ -3 \end{bmatrix}$$

(4 points) *11.* Given the coordinate matrix of vector $\bar{x}$ relative to nonstandard basis B in $\mathbb{R}^2$ to be $[\bar{x}]_B = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Find the coordinate vector of $\bar{x}$ relative to the standard basis in $\mathbb{R}^2$. $B = \{(2,4), (-1,1)\}$

$$4(2,4) - 7(-1,1) = (8,16) + (7,-7) = (15,9)$$

(7 points) *12.* Find the transition matrix P from basis B to basis C where B is the ordered basis $\{(1,-1),(3,1)\}$ in $\mathbb{R}^2$ and C is the ordered basis $\{(1,2),(-1,0)\}$.

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 3 \\ 2 & 0 & -1 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 3 \\ 0 & 2 & -3 & -5 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 3 \\ 0 & 1 & -3/2 & -5/2 \end{array}\right]$$

$$\xrightarrow{R_2+R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & -3/2 & -5/2 \end{array}\right] \qquad \begin{bmatrix} -1/2 & 1/2 \\ -3/2 & -5/2 \end{bmatrix}$$

13. Given $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1,x_2,x_3) = (2x_1 - 4x_2, x_1 + 3x_2 - 2x_3)$.

(6 points) *a.* Verify that T is a linear transformation.

$T((x_1,x_2,x_3)+(y_1,y_2,y_3)) = T(x_1+y_1, x_2+y_2, x_3+y_3) = (2(x_1+y_1) - 4(x_2+y_2), (x_1+y_1) + 3(x_2+y_2) - 2(x_3+y_3))$

$= (2x_1 + 2y_1 - 4x_2 - 4y_2, x_1 + y_1 + 3x_2 + 3y_2 - 2x_3 - 2y_3) = (2x_1 - 4x_2 + 2y_1 - 4y_2, x_1 + 3x_2 - 2x_3 + y_1 + 3y_2 - 2y_3)$

$= (2x_1 - 4x_2, x_1 + 3x_2 - 2x_3) + (2y_1 - 4y_2, y_1 + 3y_2 - 2y_3) = T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$

$T(k(x_1,x_2,x_3)) = T(kx_1,kx_2,kx_3) = (2kx_1 - 4kx_2, kx_1 + 3kx_2 - 2kx_3) = k(2x_1 - 4x_2, x_1 + 3x_2 - 2x_3) = kT(x_1,x_2,x_3)$

$\therefore$ T is a linear transformation.

(3 points) *b.* Determine the matrix of the linear transformation.

$$\begin{bmatrix} 2 & -4 & 0 \\ 1 & 3 & -2 \end{bmatrix}$$

14. Given the mapping $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ 5 & -8 & -1 \end{bmatrix}$

(6 points) a. Determine $\operatorname{Ker}(T)$.

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ 5 & -8 & -1 \end{bmatrix} \xrightarrow[-5R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 2x_2 + x_3 = 0$

$x_2 - 3x_3 = 0$ $\qquad x_1 = 5t$

$x_3 = t$

$x_2 = 3t$

$\operatorname{Ker}(T) = \{t(5,3,1) \mid t \in \mathbb{R}\}$

(3 points) b. Determine $\operatorname{Rng}(T)$.

$\operatorname{Rng}(t) = \left\{ r\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + s\begin{bmatrix} -2 \\ -3 \\ -8 \end{bmatrix} \mid r,s \in \mathbb{R} \right\}$