MATH 260 - EXAM #3 KEY

(6 pts)
#1)
$$\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & c_1+c_2 \end{bmatrix} \in S$$ $\therefore$ closed under addition

$$k\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & kc \end{bmatrix} \in S$$ $\therefore$ closed under scalar multiplication

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$$ $\therefore$ S is nonempty

$\therefore$ S = set of 2×2 upper triangular matrices is a subspace of $M_{2,2}$.

(6 pts)
#2) $S = \{(0,b,c) \mid b,c \in \mathbb{R}\}$

$(0,b_1,c_1) + (0,b_2,c_2) = (0, b_1+b_2, c_1+c_2) \in S$ $\therefore$ closed under addition

$k(0,b,c) = (0,kb,kc) \in S$ $\therefore$ closed under scalar multiplication

$(0,0,0) \in S$ $\therefore$ S is nonempty

$\therefore$ S is a subspace of $\mathbb{R}^3$.

(5 pts)
#3.) $c_1(1,2,3) + c_2(-2,0,1) + c_3(1,0,0) = (4,4,5)$

$$\begin{aligned} c_1 - 2c_2 + c_3 &= 4 \\ 2c_1 &= 4 \\ 3c_1 + c_2 &= 5 \end{aligned}$$

$c_1 = 2$

$6 + c_2 = 5$
$c_2 = -1$

$2 + 2 + c_3 = 4$
$c_3 = 0$

$2(1,2,3) - 1(-2,0,1) + 0(1,0,0) = (4,4,5)$

(5 pts)
#4) $c_1(2,0,1) + c_2(2,-1,1) + c_3(4,2,0) = (0,0,0)$

$$\rightarrow \begin{vmatrix} 2 & 2 & 4 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 1(4+4) - 1(4-0) = 8-4 = 4 \neq 0$$

$\therefore$ They are linearly independent

(5 pts)

#5.) $S = \{(4,0,1)\ (0,-3,2)\ (5,10,0)\}$

$$\begin{vmatrix} 4 & 0 & 5 \\ 0 & -3 & 10 \\ 1 & 2 & 0 \end{vmatrix} = 4(0-20) + 1(0+15) = -80+15 = -65 \neq 0$$

$\therefore$ S spans $\mathbb{R}^3$.

(5 pts)

#6.) $c_1(4,-3,6,2) + c_2(1,8,3,1) + c_3(3,-2,-1,0) = (0,0,0,0)$

$$\begin{aligned} 4c_1 + c_2 + 3c_3 &= 0 \\ -3c_1 + 8c_2 - 2c_3 &= 0 \\ 6c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 + c_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} 4 & 1 & 3 & 0 \\ -3 & 8 & -2 & 0 \\ 6 & 3 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2+R_1 \to R_1} \begin{array}{c} 3 \quad 27 \quad 3 \quad 0 \\ \begin{bmatrix} 1 & 9 & 1 & 0 \\ -3 & 8 & -2 & 0 \\ 6 & 3 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$\xrightarrow{3R_1+R_2 \to R_2} \begin{array}{c} -6 \quad -54 \quad -6 \quad 0 \\ \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 35 & 1 & 0 \\ 6 & 3 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array} \xrightarrow{-6R_1+R_3 \to R_3} \begin{array}{c} -2 \quad -18 \quad -2 \quad 0 \\ \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 35 & 1 & 0 \\ 0 & -51 & -7 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array} \xrightarrow{-2R_1+R_4 \to R_4} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 35 & 1 & 0 \\ 0 & -51 & -7 & 0 \\ 0 & -17 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 35 & 1 & 0 \\ 0 & -16 & -6 & 0 \\ 0 & -17 & -2 & 0 \end{bmatrix} \xrightarrow{-R_4} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 35 & 1 & 0 \\ 0 & -16 & -6 & 0 \\ 0 & 17 & 2 & 0 \end{bmatrix} \xrightarrow{R_3+R_4 \to R_4} \begin{array}{c} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 35 & 1 & 0 \\ 0 & -16 & -6 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \\ 0 \quad 16 \quad -64 \quad 0 \end{array}$$

$$\xrightarrow{16R_4+R_3} \begin{array}{c} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 35 & 1 & 0 \\ 0 & 0 & -70 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \\ 0 \quad -35 \quad 140 \end{array} \xrightarrow{-35R_4+R_2} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 0 & 141 & 0 \\ 0 & 0 & -70 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & -70 & 0 \\ 0 & 0 & 141 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{70}R_3} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 141 & 0 \end{bmatrix} \xrightarrow{-141R_3+R_4 \to R_4} \begin{bmatrix} 1 & 9 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} c_1 + 9c_2 + c_3 &= 0 \\ c_2 - 4c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

$$c_2 = 0$$
$$c_1 = 0$$

$\therefore$ S is linearly independent

(6pts)

#7) $S=\{1+x,\ 2x+x^2,\ 2+x+3x^2\}$

$$\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix} = 1(6-1)-1(0-2) = 1(5)-1(-2) = 5+2=7 \neq 0$$

$\therefore$ S is a basis of P_2.

(6pts)

#8.) $S=\left\{\begin{bmatrix}1&1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}\right\}$

$$\begin{array}{cccc} -1 & -1 & -1 & 0 \end{array}$$

$$\begin{vmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&1&1&1 \end{vmatrix} \underset{-1R_1+R_3}{\overset{-1R_1+R_2}{=}} \begin{vmatrix} 1&1&1&0\\ 0&0&-1&1\\ 0&-1&0&1\\ 0&1&1&1 \end{vmatrix} \overset{R_2\leftrightarrow R_4}{=} - \begin{vmatrix} 1&1&1&0\\ 0&1&1&1\\ 0&-1&0&1\\ 0&0&-1&1 \end{vmatrix} \overset{R_2+R_3}{=} - \begin{vmatrix} 1&1&1&0\\ 0&1&1&1\\ 0&0&1&2\\ 0&0&-1&1 \end{vmatrix}$$

$$\overset{R_3+R_4}{=} - \begin{vmatrix} 1&1&1&0\\ 0&1&1&1\\ 0&0&1&2\\ 0&0&0&3 \end{vmatrix} = -3 \neq 0$$

$\therefore$ S is a basis of $M_{2,2}$.

#9.) (3pts) a.) $S=\{(2,1,0)\ (1,0,0)\ (0,1,0)\}$ $\quad (2,1,0) = 2(1,0,0)+1(0,1,0)$

(2,1,0) is a linear combination of (1,0,0) and (0,1,0).

$\therefore$ S is not a basis.

(3pts) b.) $S=\{(1,4,2)\ (3,2,1)\ (0,0,0)\}$ S contains the zero vector which makes S linearly dependant

$\therefore$ S is not a basis

(3pts) c.) $S=\{(1,1,0)\ (1,0,1)\ (0,1,1)\ (1,1,1)\}$ S has 4 vectors an $\mathbb{R}^3$ basis should have 3 vectors

So S is linearly dependent. $\therefore$ S is not a basis.

(3pts) d.) $S=\{(1,2,3)\ (2,1,3)\}$ S has 2 vectors an $\mathbb{R}^3$ basis should have 3 vectors

So S does not span $\mathbb{R}^3$. $\therefore$ S is not a basis

(3pts) e.) $S=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix} \begin{bmatrix}0&1\\0&0\end{bmatrix} \begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$ S has 3 vectors an $M_{2,2}$ basis should have 4 vectors

So S does not span $M_{2,2}$ $\therefore$ S is not a basis.

#10) $A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix}$

(5pts)
a.)

$$\begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{array}{c} \begin{matrix} -5 & 15/2 & -5/2 \end{matrix} \\ \begin{bmatrix} 1 & -3/2 & 1/2 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix} \end{array} \xrightarrow{-5R_1+R_2} \begin{array}{c} \begin{matrix} -8 & 12 & -4 \end{matrix} \\ \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 35/2 & 7/2 \\ 8 & -7 & 5 \end{bmatrix} \end{array} \xrightarrow{-8R_1+R_3} \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 35/2 & 7/2 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{2}{7}R_2} \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{1/5 R_2} \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for rowspace $= \{ (1, -3/2, 1/2)\ (0, 1, 1/5) \}$

(2pts)
b.) Basis for columnspace $\left\{ \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 10 \\ -7 \end{bmatrix} \right\}$

(1 pt)
c.) $\text{rank}(A) = 2$

#11)
(5pts)
a.)

$$A = \begin{array}{c} \begin{matrix} 2 & 8 & 4 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix} \end{array} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_1 + 4c_2 + 2c_3 + c_4 = 0$

$c_2 + c_3 - c_4 = 0$

$c_4 = t$

$c_3 = s$

$c_2 = -s + t$

$c_1 = -4(t - s) - 2s - t$

$= -4t + 4s - 2s - t$

$= -5t + 2s$

$\{(-5t + 2s, -s + t, s, t) \mid s, t \in \mathbb{R}\}$

$s(2, -1, 1, 0) + t(-5, 1, 0, 1)$

(2pts)
b) basis for nullspace $= \{(2, -1, 1, 0), (-5, 1, 0, 1)\}$

(1 pt)
c.) $\text{nullity}(A) = 2.$