MATH 280 – EXAM #1
Spring Semester 2018

Name: KEY

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. You will need your Mt. SAC student identification card to submit this exam. Good luck!

(4 points) 1. Determine the distance between the points $P(3,4,-1)$ and $Q(1,5,-3)$.

$d = \sqrt{(1-3)^2 + (5-4)^2 + (-3-(-1))^2}$

$= \sqrt{(-2)^2 + (1)^2 + (-2)^2}$

$= \sqrt{4+1+4} = \sqrt{9} = 3$

(4 points) 2. Show that the given equation represents a sphere, and find its center and radius.

$$x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$$

$x^2 + 8x + 16 + y^2 - 6y + 9 + z^2 + 2z + 1 = -17 + 16 + 9 + 1$

$(x+4)^2 + (y-3)^2 + (z+1)^2 = 9$

$C(-4,3,-1)$ $r = 3$

(5 points) 3. Find the parametric equation form for the tangent line at the point $(3,0,2)$ to the curve traced by the vector function $\vec{r}(t) = (1+2\sqrt{t})i + (t^3 - t)j + (t^3 + t)\hat{k}$

^ corresponds to $t=1$

$\vec{r}'(t) = t^{-1/2}\,\hat{i} + (3t^2-1)\hat{j} + (3t^2+1)\hat{k}$

$\vec{r}'(1) = 1\hat{i} + 2\hat{j} + 4\hat{k}$

$x = 3+t$

$y = 2t$

$z = 2+4t$

(5 points) 4. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

let $x = 2\cos t$, $y = 2\sin t$

$z = (2\cos t)(2\sin t)$
$= 4\cos t\sin t$

$\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + (4\cos t\sin t)\hat{k}$

(5 points) 5. Find the equation of the plane that passes through $P(3,-1,2)$, $Q(8,2,4)$, and $R(-1,-2,-3)$.

$\vec{PQ} = \langle 5,3,2 \rangle$ $\quad \vec{PR} = \langle -4,-1,-5 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & 2 \\ -4 & -1 & -5 \end{vmatrix} = \hat{i}(-15+2) - \hat{j}(-25-8) + \hat{k}(-5+12)$$
$$= -13\hat{i} + 17\hat{j} + 7\hat{k}$$

$-13(x-3) + 17(y+1) + 7(z-2) = 0$

$-13x + 39 + 17y + 17 + 7z - 14 = 0$

$-13x + 17y + 7z + 42 = 0$

6. Given the following vectors: $\vec{a} = \langle 3, 2, -4 \rangle$, $\vec{b} = \langle -1, 5, 2 \rangle$, and $\vec{c} = \langle 3, 2, -1 \rangle$. Determine the following:

(2 points) *a.* $4\vec{a} - 3\vec{c}$

$4\langle 3,2,-4\rangle - 3\langle 3,2,-1\rangle$

$= \langle 12, 8, -16\rangle + \langle -9, -6, 3\rangle$

$= \langle 3, 2, -13\rangle$

(2 points) *b.* $|\vec{b}|$ $= \sqrt{1+25+4} = \sqrt{30}$

(3 points) *c.* $\vec{a} \cdot \vec{b}$ $= -3+10-8 = -1$

(4 points) *d.* $\vec{b} \times \vec{c}$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 5 & 2 \\ 3 & 2 & -1 \end{vmatrix} = \hat{i}(-5-4) - \hat{j}(1-6) + \hat{k}(-2-15)$

$= -9\hat{i} + 5\hat{j} - 17\hat{k}$

(2 points) *e.* The unit vector in the direction of $\vec{c}$

$|\vec{c}| = \sqrt{9+4+1} = \sqrt{14}$ $\qquad \hat{c} = \langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}} \rangle$

(2 points) *f.* The angle between vectors $\vec{a}$ and $\vec{b}$. Leave your answer in symbolic form for θ.

$|\vec{a}| = \sqrt{9+4+16} = \sqrt{29}$

$|\vec{b}| = \sqrt{1+25+4} = \sqrt{30}$ $\qquad \theta = \cos^{-1}\left(\frac{-1}{\sqrt{29}\sqrt{30}}\right)$

(4 points) *g.* $\text{proj}_{\vec{a}}\vec{b}$

$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|} = \left(\frac{-1}{29}\right)\langle 3,2,-4\rangle = \langle -\frac{3}{29}, -\frac{2}{29}, \frac{4}{29} \rangle$

(5 points) 7. Find the area of the triangle with vertices $A(0,0,-3)$, $B(4,2,0)$, and $C(3,3,1)$.

$\vec{AB} = \langle 4,2,3 \rangle$

$\vec{AC} = \langle 3,3,4 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 2 & 3 \\ 3 & 3 & 4 \end{vmatrix}$$

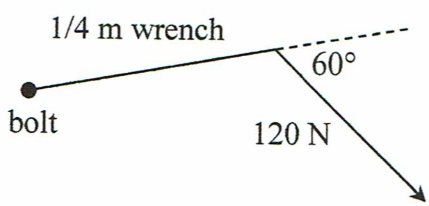
$$= \hat{\imath}(8-9) - \hat{\jmath}(16-9) + \hat{k}(12-6)$$

$$= -\hat{\imath} - 7\hat{\jmath} + 6\hat{k}$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{1+49+36} = \sqrt{86}$$

Area $\Delta = \frac{1}{2}\sqrt{86}$

(5 points) 8. A bolt is tightened by applying an 120 N force to a 1/4 meter wrench at an angle of 60° as shown below. Find the magnitude of the torque about the center of the bolt.



Torque $= |\vec{F}||\vec{D}| \sin\theta$

$= (120\text{N})(\tfrac{1}{4}\text{m}) \sin 60°$

$= (30\ \text{N}\cdot\text{m}) \frac{\sqrt{3}}{2} = 15\sqrt{3}\ \text{J}$

(4 points) 9. Explain why if $\vec{u} \cdot \vec{v} = 0$ then $\vec{u}$ and $\vec{v}$ must be orthogonal.

If $\vec{u} \cdot \vec{v} = 0 \Rightarrow |\vec{u}||\vec{v}| \cos\theta = 0$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = 90°$

$\Rightarrow$ $\vec{u}$ and $\vec{v}$ are orthogonal.

(5 points) *10.* Find the volume of the parallelepiped determined by the vectors $\vec{a} = \langle 1,2,3 \rangle$, $\vec{b} = \langle -1,1,2 \rangle$, and $\vec{c} = \langle 2,1,4 \rangle$.

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 1(4-2) - 2(-4-4) + 3(-1-2)$$

$$= 1(2) - 2(-8) + 3(-3)$$

$$= 2 + 16 - 9$$

$$= 18 - 9$$

$$= 9$$

(4 points) *11.* Evaluate the following limit: $\lim\limits_{t \to \pi/6} \vec{r}(t)$ where $\vec{r}(t) = (\sec t)\hat{i} + (\sin 2t)\hat{j} + (3t^2)\hat{k}$

$$= \left(\sec\frac{\pi}{6}\right)\hat{i} + \left(\sin\frac{\pi}{3}\right)\hat{j} + 3\left(\frac{\pi}{6}\right)^2\hat{k}$$

$$= \frac{2}{\sqrt{3}}\hat{i} + \frac{\sqrt{3}}{2}\hat{j} + \frac{\pi^2}{12}\hat{k}$$

(5 points) *12.* Evaluate the following integral: $\int_1^2 \left[(t^3)\hat{i} + \left(t\sqrt{t^2+1}\right)\hat{j} + \left(\frac{1}{t}\right)\hat{k} \right] dt$

$$= \frac{t^4}{4}\Big|_1^2 \hat{i} + \frac{1}{3}(t^2+1)^{3/2}\Big|_1^2 \hat{j} + \ln|t|\Big|_1^2 \hat{k}$$

$$= \left(\frac{16}{4} - \frac{1}{4}\right)\hat{i} + \frac{1}{3}\left(5^{3/2} - 2^{3/2}\right)\hat{j} + (\ln 2 - \ln 1)\hat{k}$$

$$= \frac{15}{4}\hat{i} + \frac{1}{3}\left(5^{3/2} - 2^{3/2}\right)\hat{j} + (\ln 2)\hat{k}$$

13. Given the position vector $\vec{r}(t) = (3\cos t)\hat{i} + (3\sin t)\hat{j} + (4t)\hat{k}$.

(5 points) a. Find the unit tangent vector $\hat{T}(t)$.

$\vec{r}'(t) = (-3\sin t)\hat{i} + (3\cos t)\hat{j} + (4)\hat{k}$

$|\vec{r}'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = 5$

$\hat{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = -\dfrac{3}{5}\sin t\,\hat{i} + \dfrac{3}{5}\cos t\,\hat{j} + \dfrac{4}{5}\hat{k}$

(5 points) b. Find the curvature κ.

$\hat{T}'(t) = -\dfrac{3}{5}\cos t\,\hat{i} - \dfrac{3}{5}\sin t\,\hat{j}$

$|\hat{T}'(t)| = \sqrt{\dfrac{9}{25}\cos^2 t + \dfrac{9}{25}\sin^2 t} = \dfrac{3}{5}$

$\kappa = \dfrac{|\hat{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{3/5}{5} = \dfrac{3}{25}$

(5 points) 14. Find the length of the curve: $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + (\ln\cos t)\hat{k}$, $0 \le t \le \pi/4$

$L = \int_0^{\pi/4} \sqrt{(-\sin t)^2 + (\cos t)^2 + \left(\frac{1}{\cos t}\cdot(-\sin t)\right)^2}\,dt$

$= \int_0^{\pi/4} \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}\,dt = \int_0^{\pi/4} \sqrt{1+\tan^2 t}\,dt = \int_0^{\pi/4} \sqrt{\sec^2 t}\,dt$

$= \int_0^{\pi/4} \sec t\,dt = \ln|\sec t + \tan t|\Big|_0^{\pi/4} = \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0|$

$= \ln|\sqrt{2}+1| - \ln|1| = \ln(\sqrt{2}+1)$

(5 points) 15. Given the vectors $\vec{u}=\langle u_1,u_2,u_3\rangle$, $\vec{v}=\langle v_1,v_2,v_3\rangle$, and $\vec{w}=\langle w_1,w_2,w_3\rangle$. Prove the following vector property: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$

$$\vec{u}+(\vec{v}+\vec{w}) = \langle u_1,u_2,u_3\rangle + (\langle v_1,v_2,v_3\rangle + \langle w_1,w_2,w_3\rangle)$$
$$= \langle u_1,u_2,u_3\rangle + \langle v_1+w_1, v_2+w_2, v_3+w_3\rangle$$
$$= \langle u_1+(v_1+w_1), u_2+(v_2+w_2), u_3+(v_3+w_3)\rangle$$
$$= \langle (u_1+v_1)+w_1, (u_2+v_2)+w_2, (u_3+v_3)+w_3\rangle$$
$$= \langle u_1+v_1, u_2+v_2, u_3+v_3\rangle + \langle w_1,w_2,w_3\rangle$$
$$= (\langle u_1,u_2,u_3\rangle + \langle v_1,v_2,v_3\rangle) + \langle w_1,w_2,w_3\rangle$$
$$= (\vec{u}+\vec{v})+\vec{w}$$

(5 points) 16. Given that $\vec{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$ where *f*, *g*, and *h* are differentiable functions of *t*. Prove that $r'(t)=f'(t)\hat{i}+g'(t)\hat{j}+h(t)\hat{k}$.

$$\vec{r}'(t) = \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{[f(t+\Delta t)\hat{i}+g(t+\Delta t)\hat{j}+h(t+\Delta t)\hat{k}]-[f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}]}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\frac{[f(t+\Delta t)-f(t)]\hat{i}+[g(t+\Delta t)-g(t)]\hat{j}+[h(t+\Delta t)-h(t)]\hat{k}}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}\hat{i} + \lim_{\Delta t\to 0}\frac{g(t+\Delta t)-g(t)}{\Delta t}\hat{j} + \lim_{\Delta t\to 0}\frac{h(t+\Delta t)-h(t)}{\Delta t}\hat{k}$$
$$= f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$$

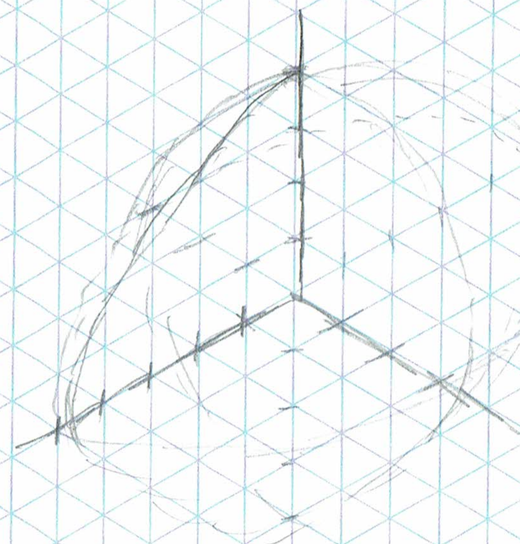
17. Graph the following surfaces on the given sheet of triangular graph paper.

(3 points) a. $\dfrac{x^2}{25}+\dfrac{y^2}{9}+\dfrac{z^2}{16}=1$

(3 points) b. $\dfrac{z}{5}=\dfrac{x^2}{9}+\dfrac{y^2}{4}$

$$\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

ellipsoid



$$\frac{z}{5} = \frac{x^2}{9} + \frac{y^2}{4}$$

paraboloid