MATH 280 – EXAM #1 **Name:** KEY

Fall Semester 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. You will need your Mt. SAC student identification card to submit this exam. Good luck!

1. Given the following points: $P(2,6,-1)$ and $Q(5,-2,1)$.

(3 points) a. Determine the distance between the points.

$$d = \sqrt{(5-2)^2+(-2-6)^2+(1-(-1))^2}$$

$$= \sqrt{(3)^2+(-8)^2+(2)^2}$$

$$= \sqrt{9+64+4}$$

$$= \sqrt{77}$$

(3 points) b. Find the equation of the line segment between P and Q.

$$\overrightarrow{PQ} = \langle 3,-8,2 \rangle$$

$$\begin{aligned} x &= 2+3t \\ y &= 6-8t \quad , \quad 0 \le t \le 1 \\ z &= -1+2t \end{aligned}$$

(5 points) 2. Find the parametric form for the equation of tangent line to the curve traced by the vector function $\vec{r}(t) = (t^2)\hat{i} + (2t-1)\hat{j} + (t^3)\hat{k}$ when $t = 2$.

$$\vec{r}\,'(t) = 2t\hat{i} + 2\hat{j} + 3t^2\hat{k}$$

$$\vec{r}\,'(2) = 4\hat{i} + 2\hat{j} + 12\hat{k}$$

vector of line tangent to curve at P $\langle 4,2,12 \rangle$

$$\vec{r}(2) = 4\hat{i} + 3\hat{j} + 8\hat{k}$$

$\hookrightarrow P(4,3,8)$

$$\begin{aligned} x &= 4+4t \\ y &= 3+2t \\ z &= 8+12t \end{aligned}$$

3. Given the following vectors: $\bar{a} = \langle 4,-1,2 \rangle$, $\bar{b} = \langle 2,1,-5 \rangle$, and $\bar{c} = \langle 1,4,-3 \rangle$. Determine the following:

(2 points) a. $4\bar{b} - 3\bar{c}$

$4\langle 2,1,-5\rangle - 3\langle 1,4,-3\rangle$

$= \langle 8,4,-20\rangle + \langle -3,-12,9\rangle$

$= \langle 5,-8,-11\rangle$

(2 points) b. $|\bar{a}|$

$\sqrt{4^2+(-1)^2+2^2}$

$= \sqrt{16+1+4}$

$= \sqrt{21}$

(3 points) c. $\bar{a}\cdot\bar{c}$

$\langle 4,-1,2\rangle \cdot \langle 1,4,-3\rangle$

$= 4+(-4)+(-6)$

$= -6$

(2 points) d. The unit vector in the direction of $\bar{c}$

$|\vec{c}| = \sqrt{1+16+9} = \sqrt{26}$

$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \langle \frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{-3}{\sqrt{26}} \rangle$

(4 points) e. $\bar{a}\times\bar{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ 2 & 1 & -5 \end{vmatrix}$$

$= \hat{i}(5-2) - \hat{j}(-20-4) + \hat{k}(4+2)$

$= 3\hat{i} + 24\hat{j} + 6\hat{k}$

(3 points) f. $\text{proj}_{\bar{b}}\bar{a}$

$\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$

$\vec{a}\cdot\vec{b} = \langle 4,-1,2\rangle \cdot \langle 2,1,-5\rangle$

$= 8-1-10 = -3$

$|\vec{b}| = \sqrt{4+1+25}$

$= \sqrt{30}$

$= \frac{-3}{30}\langle 2,1,-5\rangle$

$= -\frac{1}{10}\langle 2,1,-5\rangle$

$= \langle -\frac{1}{5}, -\frac{1}{10}, \frac{1}{2}\rangle$

(5 points) 4. Find the area of the parallelogram with vertices $A(2,4,1)$, $B(5,2,3)$, $C(7,3,0)$ and $D(4,5,-2)$.

$\vec{AB} = \langle 3,-2,2\rangle \quad \vec{AD} = \langle 2,1,-3\rangle$

$$\vec{AB}\times\vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i}(6-2) - \hat{j}(-9-4) + \hat{k}(3+4)$$

$= 4\hat{i} + 13\hat{j} + 7\hat{k}$

$|\vec{AB}\times\vec{AD}| = \sqrt{16+169+49} = \sqrt{234} = \sqrt{9\cdot 26} = 3\sqrt{26}$ square units

(5 points) 5. Find the equation of the plane that passes through $P(3,-2,5)$, $Q(2,4,1)$, and $R(-1,3,4)$.

$\vec{PQ} = \langle -1, 6, -4 \rangle$
$\vec{PR} = \langle -4, 5, -1 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 6 & -4 \\ -4 & 5 & -1 \end{vmatrix} = \hat{i}(-6+20) - \hat{j}(1-16) + \hat{k}(-5+24)$$
$$= 14\hat{i} + 15\hat{j} + 19\hat{k}$$

$14(x-3) + 15(y+2) + 19(z-5) = 0$
$14x - 42 + 15y + 30 + 19z - 95 = 0$
$14x + 15y + 19z = 107$

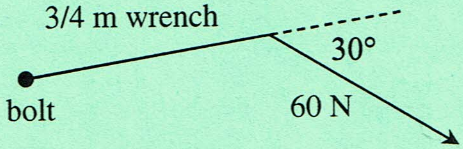
(4 points) 6. A sled is pulled along a level path by a rope. A 50-lb force acting at an angle of 60° above the horizontal moves the sled 80 feet. Find the work done by the force.

$\text{Work} = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos\theta$
$\text{Work} = (50\text{ lb})(80\text{ ft}) \cos 60°$
$= 4000 \cdot \frac{1}{2} = 2000 \text{ ft-lb.}$

(4 points) 7. Find the volume of the parallelepiped determined by the vectors $\vec{a} = \langle 3,1,2 \rangle$, $\vec{b} = \langle -2,-1,1 \rangle$, and $\vec{c} = \langle 4,2,-2 \rangle$.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 3 & 1 & 2 \\ -2 & -1 & 1 \\ 4 & 2 & -2 \end{vmatrix} = 3(2-2) - 1(4-4) + 2(-4+4)$$
$$= 3 \cdot 0 - 1 \cdot 0 + 2 \cdot 0 = 0 \text{ cubic units.}$$

(5 points) 8. A bolt is tightened by applying an 60 N force to a $3/4$ meter wrench at an angle of $30°$ as shown below. Find the magnitude of the torque about the center of the bolt.



$|T| = |\vec{r} \times \vec{D}|$

$= |\vec{F}||\vec{D}| \sin\theta$

$= (60\,N)(\tfrac{3}{4}\,m) \sin 30°$

$= \frac{45}{2}$ N-m

(3 points) 9. Evaluate the following limit: $\lim_{x\to\pi/6} \vec{r}(t)$ where $\vec{r}(t) = (\sin 2t)\hat{i} + (t^2)\hat{j} + (-\sec t)\hat{k}$

$\lim_{t\to\pi/6} \vec{r}(t) = \sin 2\frac{\pi}{6}\,\hat{i} + \left(\frac{\pi}{6}\right)^2\hat{j} + \left(-\sec\frac{\pi}{6}\right)\hat{k}$

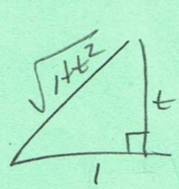
$= \frac{\sqrt{3}}{2}\hat{i} + \frac{\pi^2}{36}\hat{j} + \left(-\frac{2}{\sqrt{3}}\right)\hat{k}$

(5 points) 10. Find the length of the curve: $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + t^2\hat{k}$; $0 \le t \le \pi/4$

$\vec{r}\,'(t) = (-2\sin t)\hat{i} + (2\cos t)\hat{j} + (2t)\hat{k}$

$|\vec{r}\,'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 4t^2} = 2\sqrt{1+t^2}$

$2\int_0^{\pi/4} \sqrt{1+t^2}\,dt$ let $t = \tan\theta$, $dt = \sec^2\theta\,d\theta$



$= 2\int_0^{\pi/4} \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta$

$= 2\int_0^{\pi/4} \sec^3\theta\,d\theta = 2\left[\sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta\right] = 2\sec\tan\theta - 2\int\sec\theta(\sec^2\theta - 1)\,d\theta$

$= 2\sec\theta\tan\theta - 2\int\sec^3\theta\,d\theta + 2\int\sec\theta\,d\theta$

$u = \sec\theta$ $dv = \sec^2\theta\,d\theta$

$du = \sec\theta\tan\theta\,d\theta$ $v = \tan\theta$

$4\int\sec^3\theta\,d\theta = 2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta|$

$2\int\sec^3\theta\,d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$

$2\int_0^{\pi/4} \sec^3\theta\,d\theta$

$= \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\pi/4}$

$= \left(t\sqrt{1+t^2} + \ln\left|\sqrt{1+t^2} + t\right|\right)\Big|_0^{\pi/4}$

$= \frac{\pi}{4}\sqrt{1+\frac{\pi^2}{16}} + \ln\left|\sqrt{1+\frac{\pi^2}{16}} + \frac{\pi}{4}\right|$

11. Given the position vector: $\bar{r}(t) = (4\cos t)\hat{i} + (3t)\hat{j} + (4\sin t)\hat{k}$.

(4 points) a. Find the unit tangent vector $\hat{T}(t)$.

$\vec{r}'(t) = (-4\sin t)\hat{i} + 3\hat{j} + (4\cos t)\hat{k}$

$|\vec{r}'(t)| = \sqrt{16\sin^2 t + 9 + 16\cos^2 t} = \sqrt{16+9} = 5$

$\hat{T}(t) = -\frac{4}{5}\sin t\,\hat{i} + \frac{3}{5}\hat{j} + \frac{4}{5}\cos t\,\hat{k}$

(4 points) b. Find the curvature κ.

$\hat{T}'(t) = -\frac{4}{5}\cos t\,\hat{i} - \frac{4}{5}\sin t\,\hat{k}$

$|\hat{T}'(t)| = \sqrt{\frac{16}{25}\cos^2 t + \frac{16}{25}\sin^2 t} = \frac{4}{5}$

$\kappa = \frac{|\hat{T}'(t)|}{|\vec{r}'(t)|} = \frac{4/5}{5} = 4/25$

12. Evaluate the following integrals.

(4 points) a. $\int\left[(\tan t)\hat{i} + \left(\frac{1}{t}\right)\hat{j} + (5e^{7t})\hat{k}\right]dt$

$-\ln|\cos t|\,\hat{i} + \ln|t|\,\hat{j} + \frac{5}{7}e^{7t}\hat{k} + \vec{c}$

(4 points) *b.* $\int_0^1 \left[\left(te^t\right)\hat{i}+\left(t\sqrt{t^2+4}\right)\hat{j}+\left(\sin 2t\right)\hat{k}\right]dt$

$= \left(te^t - e^t\right)\hat{i} + \frac{1}{3}(t^2+4)^{3/2}\hat{j} + \left(-\frac{1}{2}\cos 2t\right)\hat{k}\Big|_0^1$

$= \left[(e-e)\hat{i} + \frac{1}{3}5^{3/2}\hat{j} - \frac{1}{2}\cos 2\,\hat{k}\right]$

$\quad - \left[-1\hat{i} + \frac{1}{3}(4)^{3/2}\hat{j} - \frac{1}{2}\hat{k}\right]$

$= \hat{i} + \frac{1}{3}\left(5^{3/2} - 8\right)\hat{j} + \frac{1}{2}\left(1-\cos 2\right)\hat{k}$

$\int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t$

$u = t \quad dv = e^t\,dt$

$du = dt \quad v = e^t$

$\int t\sqrt{t^2+4}\,dt \quad u = t^2+4$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$\frac{1}{2}\int u^{1/2}du$

$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}$

$= \frac{1}{3}(t^2+4)^{3/2}$

(5 points) *13.* Given the vector $\vec{u} = \langle u_1, u_2, u_3\rangle$, and scalars *c* and *d*. Prove the following vector property: $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

$(c+d)\vec{u} = (c+d)\langle u_1, u_2, u_3\rangle = \langle (c+d)u_1, (c+d)u_2, (c+d)u_3\rangle$

$= \langle cu_1 + du_1,\ cu_2 + du_2,\ cu_3 + du_3\rangle$

$= \langle cu_1, cu_2, cu_3\rangle + \langle du_1, du_2\, du_3\rangle$

$= c\langle u_1, u_2, u_3\rangle + d\langle u_1, u_2, u_3\rangle$

$= c\vec{u} + d\vec{u}$

(5 points) *14.* Given the vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$, and $\vec{w} = \langle w_1, w_2, w_3\rangle$. Prove the following vector property: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

$\vec{u} + (\vec{v} + \vec{w}) = \langle u_1, u_2, u_3\rangle + (\langle v_1, v_2, v_3\rangle + \langle w_1, w_2, w_3\rangle)$

$= \langle u_1, u_2, u_3\rangle + (\langle v_1 + w_1,\ v_2 + w_2,\ v_3 + w_3\rangle)$

$= \langle u_1 + (v_1 + w_1),\ u_2 + (v_2 + w_2),\ u_3 + (v_3 + w_3)\rangle$

$= \langle (u_1 + v_1) + w_1,\ (u_2 + v_2) + w_2,\ (u_3 + v_3) + w_3\rangle$

$= \langle u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3\rangle + \langle w_1, w_2, w_3\rangle$

$= (\langle u_1, u_2, u_3\rangle + \langle v_1, v_2, v_3\rangle) + \langle w_1, w_2, w_3\rangle$

$= (\vec{u} + \vec{v}) + \vec{w}$

(5 points) *15.* Prove the following: If $|\vec{r}(t)| = c$, then $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal for all t.

Let $|\vec{r}(t)| = c$

Then $|\vec{r}(t)|^2 = c^2$

so $|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t) = c^2$

$\frac{d}{dt}\vec{r}(t)\cdot\vec{r}(t) = \frac{d}{dt}c^2$

$\vec{r}'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}'(t) = 0$

$2[\vec{r}(t)\cdot\vec{r}'(t)] = 0$

$\vec{r}(t)\cdot\vec{r}'(t) = 0$

$\therefore$ $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.

(5 points) *16.* Given that $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ where f, g, and h are differentiable functions of t. Prove that $r'(t) = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$.

$$\vec{r}'(t) = \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{[f(t+\Delta t)\hat{i} + g(t+\Delta t)\hat{j} + h(t+\Delta t)\hat{k}] - [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}]}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{[f(t+\Delta t) - f(t)]\hat{i} + [g(t+\Delta t) - g(t)]\hat{j} + [h(t+\Delta t) - h(t)]\hat{k}}{\Delta t}$$

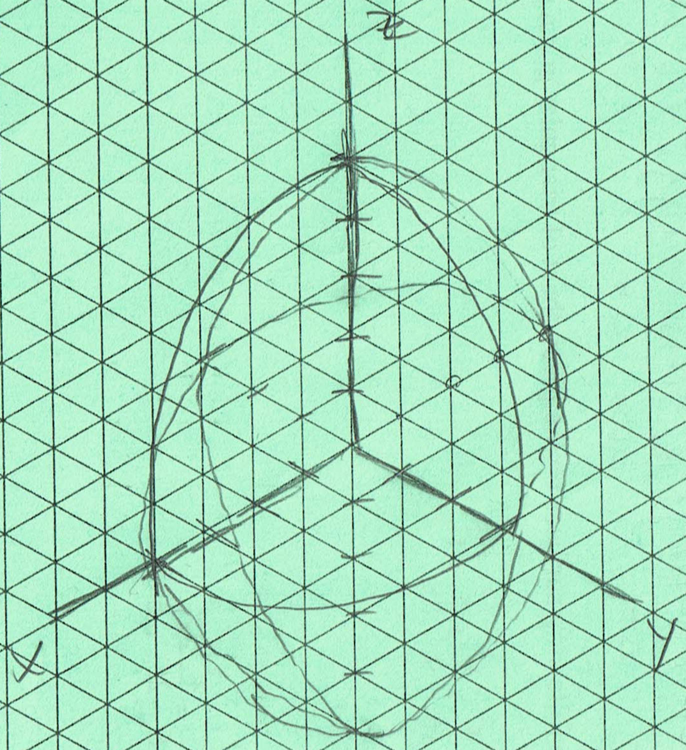
$$= \lim_{\Delta t\to 0}\left(\frac{f(t+\Delta t) - f(t)}{\Delta t}\right)\hat{i} + \lim_{\Delta t\to 0}\frac{g(t+\Delta t) - g(t)}{\Delta t}\Big)\hat{j} + \lim_{\Delta t\to 0}\frac{h(t+\Delta t) - h(t)}{\Delta t}\hat{k} = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$$

17. Graph the following surfaces on the given sheet of triangular graph paper.

(4 points) *a.* $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{25} = 1$

(4 points) *b.* $\frac{z}{6} = \frac{x^2}{25} + \frac{y^2}{16}$

$$\frac{x^2}{16}+\frac{y^2}{9}+\frac{z^2}{25}=1$$



$$\frac{z}{6}=\frac{x^2}{25}+\frac{y^2}{16}$$