MATH 280 – EXAM #1
Winter Session 2020

Name: Key

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may not use a calculator on this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You will need your Mt. SAC student identification card to submit this exam. Good luck!

(4 points) 1. Determine the distance between the points $P(4,9,-5)$ and $Q(8,7,-2)$.

$d=\sqrt{(8-4)^2+(7-9)^2+(-2-(-5))^2}$

$=\sqrt{(4)^2+(-2)^2+(3)^2}$

$=\sqrt{16+4+9}$

$=\sqrt{29}$

(4 points) 2. Show that the given equation represents a sphere, and find its center and radius.

$$x^2+y^2+z^2-4x+2y-6z-11=0$$

$x^2-4x+4+y^2+2y+1+z^2-6z+9=11+4+1+9$

$(x-2)^2+(y+1)^2+(z-3)^2=25$

center $(2,-1,3)$

radius 5

(5 points) 3. Find the parametric equation form for the tangent line to the curve traced by the vector function $\vec{r}(t) = (3+t)\hat{i} + (e^{2t})\hat{j} + \left(\sqrt{t^2+1}\right)\hat{k}$ when $t = 0$.

$\vec{r}\,'(t) = \hat{i} + 2e^{2t}\hat{j} + \frac{t}{\sqrt{t^2+1}}\hat{k}$

$\vec{r}\,'(0) = \hat{i} + 2\hat{j} + 0\hat{k}$ $\quad\vec{r}(0) = 3\hat{i} + \hat{j} + \hat{k}$ point at $t=0$ $(3,1,1)$

$x = 3 + t$

$y = 1 + 2t$

$z = 1$

(5 points) 4. Find the equation of the plane that passes through $P(8,3,-4)$, $Q(5,2,-6)$, and $R(4,7,-1)$.

$\vec{PQ} = \langle -3,-1,-2\rangle$

$\vec{PR} = \langle -4,4,3\rangle$

$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -1 & -2 \\ -4 & 4 & 3 \end{vmatrix} = \hat{i}(-3+8) - \hat{j}(-9-8) + \hat{k}(-12-4)$

$= 5\hat{i} + 17\hat{j} - 16\hat{k}$

$5(x-8) + 17(y-3) - 16(z+4) = 0$

$5x - 40 + 17y - 51 - 16z - 64 = 0$

$5x + 17y - 16z - 155 = 0$

(4 points) 5. A sled is pulled along a level path by a rope. A 40-lb force acting at an angle of 30° above the horizontal moves the sled a total distance of 80 feet. Find the work done by the force.

$\text{Work} = \vec{F}\cdot\vec{D} = |\vec{F}||\vec{D}|\cos\theta$

$= (40\,\text{lb})(80\,\text{ft})\cos 30^\circ$

$= 3200\frac{\sqrt{3}}{2}$ ft-lb

$= 1600\sqrt{3}$ ft-lb.

6. Given the following vectors: $\vec{a}=\langle 6,2,-3\rangle$, $\vec{b}=\langle -2,3,-1\rangle$, and $\vec{c}=\langle 4,2,-1\rangle$. Determine the following:

(2 points) a. $4\vec{a}+2\vec{b}$ $= 4\langle 6,2,-3\rangle + 2\langle -2,3,-1\rangle$

$= \langle 24,8,-12\rangle + \langle -4,6,-2\rangle$

$= \langle 20,14,-14\rangle$

(2 points) b. $|\vec{c}|$ $= \sqrt{16+4+1} = \sqrt{21}$

(3 points) c. $\vec{a}\cdot\vec{c}$ $= 24+4+3 = 31$

(4 points) d. $\vec{a}\times\vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 2 & -3 \\ -2 & 3 & -1 \end{vmatrix} = \hat{i}(-2+9) - \hat{j}(-6-6) + \hat{k}(18+4)$$

$= 7\hat{i} + 12\hat{j} + 22\hat{k}$

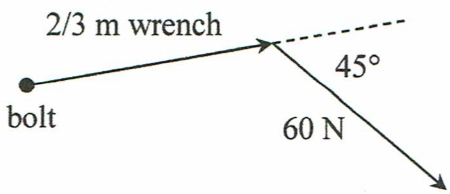
(2 points) e. The unit vector in the direction of $\vec{c}$ $\dfrac{\vec{c}}{|\vec{c}|} = \dfrac{\langle 4,2,-1\rangle}{\sqrt{21}}$

$= \left\langle \dfrac{4}{\sqrt{21}}, \dfrac{2}{\sqrt{21}}, \dfrac{-1}{\sqrt{21}} \right\rangle$

(4 points) f. $\text{proj}_{\vec{a}}\vec{c}$ $= \left(\dfrac{\vec{a}\cdot\vec{c}}{|\vec{a}|}\right)\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{31}{49}\langle 6,2,-3\rangle = \left\langle \dfrac{186}{49}, \dfrac{62}{49}, \dfrac{-93}{49} \right\rangle$

$|\vec{a}| = \sqrt{36+4+9} = 7$

(4 points) 8. A bolt is tightened by applying an 60 N force to a 2/3 meter wrench at an angle of 45° as shown below. Find the magnitude of the torque about the center of the bolt.



$$|\vec{\tau}| = |\vec{r} \times \vec{F}|$$
$$= |\vec{r}||\vec{F}| \sin\theta$$
$$= \left(\tfrac{2}{3} m\right)(60 N) \sin 45°$$
$$= 40 \frac{\sqrt{2}}{2} N\text{-}m$$
$$= 20\sqrt{2}\ N\text{-}m$$

(4 points) 9. Prove that if $\vec{u} \cdot \vec{v} = 0$ then $\vec{u}$ and $\vec{v}$ must be orthogonal.

$$\vec{u} \cdot \vec{v} = 0 \Rightarrow |\vec{u}||\vec{v}| \cos\theta = 0$$
$$\cos\theta = 0$$
$$\theta = \pi/2$$

$\therefore$ $\vec{u}$ and $\vec{v}$ are orthogonal.

(4 points) 10. Find the volume of the parallelepiped determined by the vectors $\vec{a} = \langle 4, -2, 1 \rangle$, $\vec{b} = \langle 2, 5, 0 \rangle$, and $\vec{c} = \langle 3, 2, -2 \rangle$.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 4 & -2 & 1 \\ 2 & 5 & 0 \\ 3 & 2 & -2 \end{vmatrix} = 4(-10-0) - (-2)(-4-0) + 1(4-15)$$
$$= 4(-10) + 2(-4) + 1(-11)$$
$$= -40 - 8 - 11 = -59$$

Volume $= 59$ cubic units.

(4 points) 11. Evaluate the following limit: $\lim\limits_{t\to\pi/6}\vec{r}(t)$ where $\vec{r}(t)=(3t^2)\hat{i}+(\sin 4t)\hat{j}+(\sec t)\hat{k}$

$$\lim_{t\to\pi/6}\vec{r}(t)=3\left(\frac{\pi}{6}\right)^2\hat{i}+\sin 4\left(\frac{\pi}{6}\right)\hat{j}+\sec\left(\frac{\pi}{6}\right)\hat{k}$$

$$=\frac{\pi^2}{12}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}+\frac{2}{\sqrt{3}}\hat{k}$$

(5 points) 12. Evaluate the following integral: $\int_1^2\left[(t^2)\hat{i}+\left(\sqrt[3]{t-1}\right)\hat{j}+\left(\frac{1}{t^2}\right)\hat{k}\right]dt$

$$=\left(\frac{t^3}{3}\hat{i}+\frac{3}{4}(t-1)^{4/3}\hat{j}+\left(-\frac{1}{t}\right)\hat{k}\right)\Big|_1^2$$

$$=\left(\frac{8}{3}\hat{i}+\frac{3}{4}\hat{j}-\frac{1}{2}\hat{k}\right)-\left(\frac{1}{3}\hat{i}-\hat{k}\right)$$

$$=\frac{7}{3}\hat{i}+\frac{3}{4}\hat{j}+\frac{1}{2}\hat{k}$$

(5 points) 13. Evaluate the following integral: $\int\left[(\cot 3t)\hat{i}+(4\cos 2t)\hat{j}+(te^{2t})\hat{k}\right]dt$

$$\frac{1}{3}\ln|\sin 3t|\hat{i}+(2\sin 2t)\hat{j}+\left(\frac{1}{2}te^{2t}-\frac{1}{4}e^{2t}\right)\hat{k}+\vec{C}$$

$\int te^{2t}dt$

$u=t \quad dv=e^{2t}dt$

$du=dt \quad v=\frac{1}{2}e^{2t}$

$\int te^{2t}=\frac{1}{2}te^{2t}-\frac{1}{2}\int e^{2t}dt$

$=\frac{1}{2}te^{2t}-\frac{1}{4}e^{2t}$

14. Given the position vector $\vec{r}(t) = (6t)\hat{i} + (4\cos 2t)\hat{j} + (4\sin 2t)\hat{k}$.

(4 points) a. Find the unit tangent vector, $\hat{T}(t)$.

$\vec{r}'(t) = 6\hat{i} + (-8\sin 2t)\hat{j} + (8\cos 2t)\hat{k}$

$|\vec{r}'(t)| = \sqrt{36 + 64\sin^2 2t + 64\cos^2 2t} = 10$

$\hat{T}(t) = \frac{3}{5}\hat{i} + (-\frac{4}{5}\sin 2t)\hat{j} + (\frac{4}{5}\cos 2t)\hat{k}$

(4 points) b. Find the unit normal vector $\hat{N}(t)$.

$\hat{T}'(t) = (-\frac{8}{5}\cos 2t)\hat{j} + (-\frac{8}{5}\sin 2t)\hat{k}$

$|\hat{T}'(t)| = \sqrt{\frac{64}{25}\cos^2 2t + \frac{64}{25}\sin^2 2t} = \frac{8}{5}$

$\hat{N}(t) = (-\cos 2t)\hat{j} + (-\sin 2t)\hat{k}$

(4 points) c. Find the binormal vector $\hat{B}(t)$.

$$\hat{B}(t) = \hat{T} \times \hat{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5} & -\frac{4}{5}\sin 2t & \frac{4}{5}\cos 2t \\ 0 & -\cos 2t & -\sin 2t \end{vmatrix} = \hat{i}\left(\frac{4}{5}\sin^2 2t + \frac{4}{5}\cos^2 2t\right)\hat{i} - \hat{j}\left(-\frac{3}{5}\sin 2t\right) + \hat{k}\left(-\frac{3}{5}\cos 2t\right)$$

$$= \frac{4}{5}\hat{i} + \left(\frac{3}{5}\sin 2t\right)\hat{j} + \left(-\frac{3}{5}\cos 2t\right)\hat{k}$$

(4 points) d. Find the curvature κ.

$$\kappa = \frac{|\hat{T}'(t)|}{|\vec{r}'(t)|} = \frac{8/5}{10} = \frac{8}{5} \cdot \frac{1}{10} = \frac{4}{25}$$

(5 points) *15.* Find the length of the curve: $\vec{r}(t)=(2t)\hat{i}+(t^2)\hat{j}+\left(\frac{1}{3}t^3\right)\hat{k};\quad 0\le t\le 1$

$$L=\int_0^1\sqrt{(2)^2+(2t)^2+(t^2)^2}\,dt=\int_0^1\sqrt{4+4t^2+t^4}\,dt$$

$$=\int_0^1\sqrt{(t^2+2)^2}\,dt=\int_0^1(t^2+2)\,dt=\left(\frac{t^3}{3}+2t\right)\Big|_0^1=\frac{1}{3}+2=\frac{7}{3}$$

(5 points) *16.* Given the vector $\vec{u}=\langle u_1,u_2,u_3\rangle$ and scalars *c* and *d*. Prove the following vector property: $(c+d)\vec{u}=c\vec{u}+d\vec{u}$

$$(c+d)\vec{u}=(c+d)\langle u_1,u_2,u_3\rangle=\langle(c+d)u_1,(c+d)u_2,(c+d)u_3\rangle$$

$$=\langle cu_1+du_1,\ cu_2+du_2,\ cu_3+du_3\rangle$$

$$=\langle cu_1,cu_2,cu_3\rangle+\langle du_1,du_2,du_3\rangle$$

$$=c\langle u_1,u_2,u_3\rangle+d\langle u_1,u_2,u_3\rangle$$

$$=c\vec{u}+d\vec{u}$$

(5 points) *17.* Given that $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ where *f*, *g*, and *h* are differentiable functions of *t*. Prove that $r'(t) = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$.

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{[f(t+\Delta t)\hat{i} + g(t+\Delta t)\hat{j} + h(t+\Delta t)\hat{k}] - [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}]}{\Delta t}$$

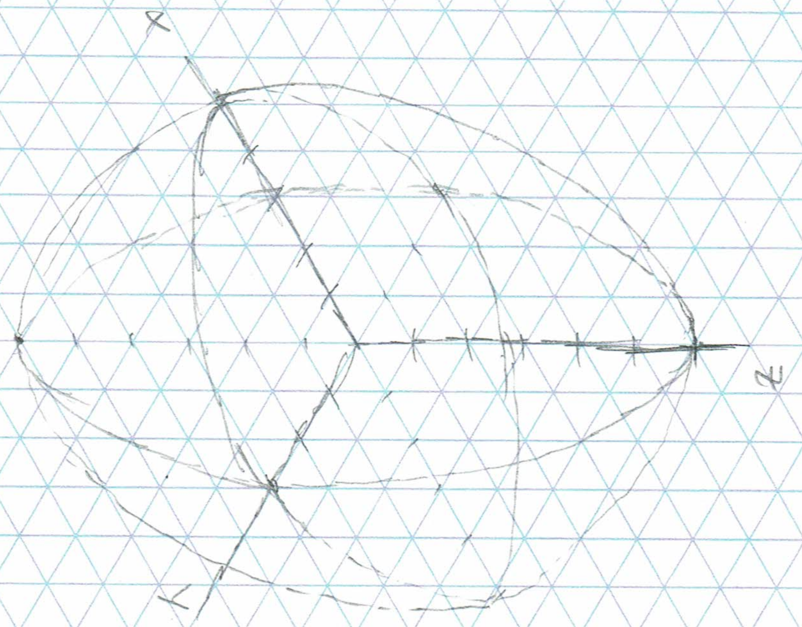
$$= \lim_{\Delta t \to 0} \frac{[f(t+\Delta t) - f(t)]}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{[g(t+\Delta t) - g(t)]}{\Delta t}\hat{j} + \lim_{\Delta t \to 0} \frac{[h(t+\Delta t) - h(t)]}{\Delta t}\hat{k}$$

$$= f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$$

18. Graph the following surfaces on the given sheet of triangular graph paper.

(3 points) *a.* $\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{36} = 1$

(3 points) *b.* $\frac{z}{6} = \frac{x^2}{16} + \frac{y^2}{4}$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