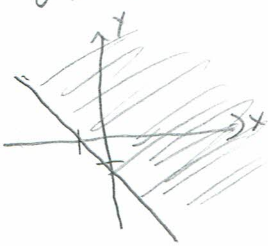
Exam#2

(4pts) 1.) $f(x,y)=\sqrt{y+x+1}$

$\{(x,y) \mid y+x+1 \geq 0\}$

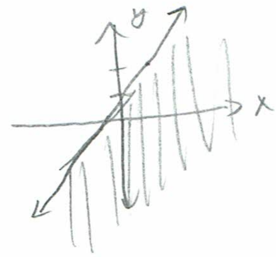
$y \geq -x-1$



$f(x,y)=\sqrt{x-y+1}$

$\{(x,y) \mid x-y+1 \geq 0\}$

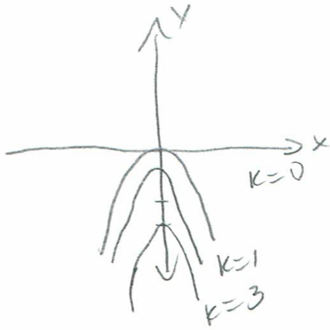
$x+1 \geq y$



(4pts) 2) $f(x,y)=x^2+y$ $k=0, k=1, k=3$

$x^2+y=0$ $x^2+y=1$ $x^2+y=3$

$y=-x^2$ $y=1-x^2$ $y=3-x^2$



(4pts) 3.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2-y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^2-y^2)}{x^2-y^2}$$

$$= \lim_{(x,y)\to(0,0)} (x^2+y^2) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^6-y^6}{x^3-y^3}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^3+y^3)(x^3-y^3)}{x^3-y^3}$$

$$= \lim_{(x,y)\to(0,0)} (x^3+y^3) = 0$$

(4pts) 4.) $\lim\limits_{(x,y)\to(3,6)} \dfrac{x+y-9}{\sqrt{x+y}-3}$

$= \lim\limits_{(x,y)\to(3,6)} \dfrac{x+y-9}{\sqrt{x+y}-3} \cdot \dfrac{\sqrt{x+y}+3}{\sqrt{x+y}+3}$

$= \lim\limits_{(x,y)\to(3,6)} \dfrac{(x+y-9)(\sqrt{x+y}+3)}{x+y-9}$

$= \lim\limits_{(x,y)\to(3,6)} (\sqrt{x+y}+3) = 6$

$\lim\limits_{(x,y)\to(16,9)} \dfrac{x+y-25}{\sqrt{x+y}-5}$

$= \lim\limits_{(x,y)\to(16,9)} \dfrac{x+y-25}{\sqrt{x+y}-5} \cdot \dfrac{\sqrt{x+y}+5}{\sqrt{x+y}+5}$

$= \lim\limits_{(x,y)\to(16,9)} \dfrac{(x+y-25)(\sqrt{x+y}+5)}{x+y-25}$

$= \lim\limits_{(x,y)\to(16,9)} (\sqrt{x+y}+5) = 10$

(4pts) 5.) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^4+y^4}$

$y=0$ $\quad \lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \dfrac{x^4-y^4}{x^4+y^4} = \lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \dfrac{x^4}{x^4} = 1$

$x=0$ $\quad \lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} \dfrac{x^4-y^4}{x^4+y^4} = \lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} \left(\dfrac{-y^4}{y^4}\right) = -1$

Since the limits are different, by the two-paths approach, the limit does not exist.

6.) $f(x,y) = 4x^3\sin(x^4y^2)$

(2pts) a.) $f_x = 12x^2\sin(x^4y^2) + 4x^3\, 4x^3y^2\cos(x^4y^2)$

$= 12x^2\sin(x^4y^2) + 16x^6y^2\cos(x^4y^2)$

(2pts) b.) $f_y = 4x^3\, 2x^4y\cos(x^4y^2) = 8x^7y\cos(x^4y^2)$

(3pts) c.) $f_{xx} = 24x\sin(x^4y^2) + 12x^2\, 4x^3y^2\cos(x^4y^2)$

$+ 96x^5y^2\cos(x^4y^2) - 16x^6y^2\, 4x^3y^2\sin(x^4y^2)$

$= 24x\sin(x^4y^2) + 144x^5y^2\cos(x^4y^2) - 64x^9y^4\sin(x^4y^2)$

(3pts) d.) $f_{xy} = 12x^2\, 2x^4y\cos(x^4y^2) + 32x^6y\cos(x^4y^2) - 16x^6y^2\, 2x^4y\sin(x^4y^2)$

$= 24x^6y\cos(x^4y^2) + 32x^6y\cos(x^4y^2) - 32x^{10}y^3\sin(x^4y^2)$

(3pts) e.) $f_{yy} = 8x^7\cos(x^4y^2) - 8x^7y\, 2x^4y\sin(x^4y^2)$

$= 8x^7\cos(x^4y^2) - 16x^{11}y^2\sin(x^4y^2)$

$f(x,y) = 4y^2\cos(x^3y^4)$

a.) $f_x = -4y^2\, 3x^2y^4\sin(x^3y^4) = -12x^2y^6\sin(x^3y^4)$

b.) $f_y = 8y\cos(x^3y^4) - 4y^2\, 4x^3y^3\sin(x^3y^4)$

$= 8y\cos(x^3y^4) - 16x^3y^5\sin(x^3y^4)$

c.) $f_{xx} = -24xy^6\sin(x^3y^4) - 12x^2y^6\, 3x^2y^4\cos(x^3y^4)$

$= -24xy^6\sin(x^3y^4) - 36x^4y^{10}\cos(x^3y^4)$

d.) $f_{xy} = -72x^2y^5\sin(x^3y^4) - 12x^2y^6\, 4x^3y^3\cos(x^3y^4)$

$= -72x^2y^5\sin(x^3y^4) - 48x^5y^9\cos(x^3y^4)$

e.) $f_{yy} = 8\cos(x^3y^4) - 8y\, 4x^3y^3\sin(x^3y^4)$

$-80x^3y^4\sin(x^3y^4) - 16x^3y^5\, 4x^3y^3\cos(x^3y^4)$

$= 8\cos(x^3y^4) - 112x^3y^4\sin(x^3y^4)$

$-64x^6y^8\cos(x^3y^4)$

(4pts) 7) $z=x^3+y^2,\ x=3s^4+5t^3,\ y=6s^2+7t^5$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$=3x^2\,12s^3+2y\,12s$$

$$=36x^2s^3+24ys$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$=3x^2\,15t^2+2y\,35t^4$$

$$=45x^2t^2+70yt^4$$

$z=x^4+y^3,\ x=5s^3+4t^2,\ y=3s^5+6t^4$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$=4x^3\,15s^2+3y^2\,15s^4$$

$$=60x^3s^2+45y^2s^4$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$=4x^3\,8t+3y^2\,24t^3$$

$$=32x^3t+72t^3y^2$$

(4pts) 8) $4y^3+e^{x^3y^2}=5x^4y^3+6x^2$

$$4y^3+e^{x^3y^2}-5x^4y^3-6x^2=0$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

$$\frac{dy}{dx}=-\frac{3x^2y^2e^{x^3y^2}-20x^3y^3-12x}{12y^2+2x^3y\,e^{x^3y^2}-15x^4y^2}$$

$$3x^4+e^{x^3y^2}=4x^3y^5+7x^3$$

$$3x^4+e^{x^3y^2}-4x^3y^5-7x^3=0$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

$$\frac{dy}{dx}=-\frac{12x^3+3x^2y^2e^{x^3y^2}-12x^2y^5-21x^2}{2x^3y\,e^{x^3y^2}-20x^3y^4}$$

9. $f(x,y)=y^2e^{xy}$

(4pts) a. $f_x=y^2\,y\,e^{xy}=y^3e^{xy}$

$f_y=2y\,e^{xy}+y^2x\,e^{xy}$

$f_x(0,2)=8 \qquad f_y(0,2)=4 \qquad f(0,2)=4$

$$z-4=8(x-0)+4(y-2)$$

$$z-4=8x+4y-8$$

$$z=8x+4y-4$$

$$L(x,y)=8x+4y-4$$

(2pts) b. $L(0.2,2.1)$

$$=8(0.2)+4(2.1)-4$$

$$=1.6+8.4-4$$

$$=6$$

(4pts) 10.) $f(x,y,z) = x^2 y + x\sqrt{1+z}$

$\vec{\nabla} f = (2xy + \sqrt{1+z})\hat{\imath} + x^2 \hat{\jmath} + x\frac{1}{2}(1+z)^{-1/2}\hat{k}$

$\vec{\nabla} f(1,2,3) = (4+2)\hat{\imath} + 1\hat{\jmath} + \frac{1}{2}\cdot\frac{1}{2}\hat{k}$

$= 6\hat{\imath} + \hat{\jmath} + \frac{1}{4}\hat{k}$

$\vec{v} = \langle 2,1,-2 \rangle$

$|\vec{v}| = \sqrt{4+1+4} = 3$

$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2,1,-2\rangle}{3} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle$

$D_{\hat{v}} f = (6\hat{\imath} + \hat{\jmath} + \frac{1}{4}\hat{k})\cdot(\frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k})$

$= 4 + \frac{1}{3} - \frac{1}{6} = \frac{24+2-1}{6} = \frac{25}{6}$

11.) $xy^2z^3 = 8 \quad P_0(2,2,1)$

(5pts) a.

$\vec{\nabla} f = (y^2z^3)\hat{\imath} + (2xyz^3)\hat{\jmath} + (3xy^2z^2)\hat{k}$

$\vec{\nabla} f(2,2,1) = 4\hat{\imath} + 8\hat{\jmath} + 24\hat{k}$

$4(x-2) + 8(y-2) + 24(z-1) = 0$

$4x - 8 + 8y - 16 + 24z - 24 = 0$

$4x + 8y + 24z - 48 = 0$

(2pts) b.)

$x = 2 + 4t$

$y = 2 + 8t$

$z = 1 + 24t$

12) $f(x,y) = x^2y + \sqrt{y} \quad P_0(2,1)$

(2pts) a.) $\vec{\nabla} f = (2xy)\hat{\imath} + (x^2 + \frac{1}{2}y^{-1/2})\hat{\jmath}$

$\vec{\nabla} f(2,1) = 4\hat{\imath} + \frac{9}{2}\hat{\jmath}$

(2pts) b) $-\vec{\nabla} f = -4\hat{\imath} - \frac{9}{2}\hat{\jmath}$

(2pts) c.) $\vec{z}_1 = -\frac{9}{2}\hat{\imath} + 4\hat{\jmath}$

$\vec{z}_2 = \frac{9}{2}\hat{\imath} - 4\hat{\jmath}$

(7pts.) 13.) $f(x,y) = x^3 - 6xy + 8y^3$

$f_x = 3x^2 - 6y$

$3x^2 - 6y = 0$

$3x^2 = 6y$

$x^2 = 2y$

$f_y = -6x + 24y^2$

$-6x + 24y^2 = 0$

$24y^2 = 6x$

$4y^2 = x$

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0,\ x = 1$

$x = 0$ $\quad x^2 = 2y$

$0 = 2y$

$y = 0$ $\quad (0,0)$

$x = 1$ $\quad 1 = 2y$ $\quad (1, 1/2)$

$y = 1/2$

$f_{xx} = 6x \quad f_{yy} = 48y \quad f_{xy} = -6$

$D(0,0) = (0)(0) - (-6)^2 = -36 < 0$ $\quad (0,0)$ saddle point

$D(1, 1/2) = (6)(24) - (-6)^2 > 0$ $\quad f_{xx} = 6 > 0$ local minimum

(7pts) 14) $f(x,y,z) = xyz$ $\quad g(x,y,z) = x^2 + y^2 + z^2 = 3$

$\vec{\nabla} f = (yz)\hat{i} + (xz)\hat{j} + (xy)\hat{k}$ $\quad \vec{\nabla} g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$yz = \lambda 2x \quad xz = \lambda 2y \quad xy = \lambda 2z \quad x^2 + y^2 + z^2 = 3$

$\frac{yz}{x} = 2\lambda \quad xz = \frac{yz}{x} y \quad xy = \left(\frac{yz}{x}\right) z \quad x^2 + x^2 + x^2 = 3$

$(x \neq 0)$

$x^2 = y^2 \quad x^2 = z^2$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$x=1 \ y=1 \ z=1$ $\quad f(1,1,1) = 1$

$x=1 \ y=1 \ z=-1$ $\quad f(1,1,-1) = -1$

$x=1 \ y=-1 \ z=1$ $\quad f(1,-1,1) = -1$

$x=1 \ y=-1 \ z=-1$ $\quad f(1,-1,-1) = 1$

$x=-1 \ y=1 \ z=1$ $\quad f(-1,1,1) = -1$

$x=-1 \ y=1 \ z=-1$ $\quad f(-1,1,-1) = 1$

$x=-1 \ y=-1 \ z=1$ $\quad f(-1,-1,1) = 1$

$x=-1 \ y=-1 \ z=-1$ $\quad f(-1,-1,-1) = -1$

maximum value = 1

minimum value = -1.

(7 pts) 15) $V = xyz$ $\quad 2xy + 2xz + 2yz = 64$

$\vec{\nabla} V = (yz)\hat{\imath} + (xz)\hat{\jmath} + (xy)\hat{k}$ $\quad \vec{\nabla} g = (2y+2z)\hat{\imath} + (2x+2z)\hat{\jmath} + (2x+2y)\hat{k}$

$yz = \lambda(2y+2z)$ $\quad xz = \lambda(2x+2z)$ $\quad xy = \lambda(2x+2y)$ $\quad 2xy+2xz+2yz = 64$

① $yz = 2\lambda y + 2\lambda z$ $\quad$ ② $xz = 2\lambda x + 2\lambda z$ $\quad$ ③ $xy = 2\lambda x + 2\lambda y$

① & ②

$\boxed{yz - 2\lambda y} = 2\lambda z$ $\quad xz - 2\lambda x = \boxed{2\lambda z}$

$xz - 2\lambda x = yz - 2\lambda y$

$x(z - 2\lambda) = y(z - 2\lambda)$

$x = y$

① & ③

$\boxed{yz - 2\lambda z} = 2\lambda y$ $\quad xy - 2\lambda x = \boxed{2\lambda y}$

$xy - 2\lambda x = yz - 2\lambda z$

$x(y - 2\lambda) = z(y - 2\lambda)$

$x = z$

$2x^2 + 2x^2 + 2x^2 = 64$

$6x^2 = 64$

$x^2 = \frac{32}{3}$

$x = \sqrt{\frac{32}{3}} = \frac{4\sqrt{2}}{\sqrt{3}}$ cm $\quad y = \frac{4\sqrt{2}}{\sqrt{3}}$ cm $\quad z = \frac{4\sqrt{2}}{\sqrt{3}}$ cm

Volume $= \left(\frac{4\sqrt{2}}{\sqrt{3}}\right)\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)\left(\frac{4\sqrt{2}}{\sqrt{3}}\right) = \frac{64(2)}{(3)}\frac{\sqrt{2}}{\sqrt{3}} = \frac{128\sqrt{2}}{9}$ cm^3

(4 pts EC) 16.) $f(x,y) = x^2 - 4y^2$ $\quad P_0(2,-1)$ $\quad \hat{u} = \frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$

$\lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h} = \lim_{h\to 0} \frac{(2+\frac{3}{5}h)^2 - 4(-1-\frac{4}{5}h)^2 - 0}{h}$

$= \lim_{h\to 0} \frac{4 + \frac{12}{5}h + \frac{9}{25}h^2 - 4(1 + \frac{8}{5}h + \frac{16}{25}h^2)}{h}$

$= \lim_{h\to 0} \frac{4 + \frac{12}{5}h + \frac{9}{25}h^2 - 4 - \frac{32}{5}h - \frac{64}{25}h^2}{h} = \lim_{h\to 0} \frac{-4h - \frac{55}{25}h^2}{h} = \lim_{h\to 0} \frac{h(-4 - \frac{55}{25}h)}{h}$

$= \lim_{h\to 0} (-4 - \frac{55}{25}h) = -4$