Directions: *Please show all work for maximum credit.* *No work = no credit.* Point values for each problem are given. There are 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. Good luck!

1. Given that the following position vector describes a projectile in motion where distance is in feet and time is in seconds: $\vec{r}(t) = \left(32\sqrt{3}\right)t\hat{i} + \left(128 + 32t - 16t^2\right)\hat{j}$

(3 points) a. What is the maximum height that the projectile attains?

$$v_y = 128 + 32t - 16t^2$$
$$v_y' = 32 - 32t = 0$$
$$32 = 32t$$
$$t = 1 \text{ sec}$$

$$v_y(1) = 128 + 32(1) - 16(1)^2$$
$$= 128 + 32 - 16$$
$$= 144 \text{ ft}$$

(3 points) b. When does the projectile hit the ground?

$$v_y = 128 + 32t - 16t^2 = 0$$
$$-16(t^2 - 2t - 8) = 0$$
$$-16(t - 4)(t + 2) = 0$$
$$t - 4 = 0 \qquad t + 2 = 0$$
$$t = 4 \text{ sec} \qquad t = -2$$
$$\qquad\qquad\qquad \times$$

(3 points) c. What is the range of the projectile?

$$v_x(4) = 32\sqrt{3}(4) = 128\sqrt{3} \text{ ft}$$

(3 points) 2. A baseball is thrown at a speed of 15 ft/s from the stands 20 ft above the field at an angle of 30° up from the horizontal. At the instant the ball is thrown, an instantaneous horizontal gust of wind of 4 ft/s blows with the direction in which the ball is thrown. What is the position vector that models this situation? $\left(g = 32 \ \text{ft/s}^2\right)$

$$\vec{r}(t) = (15\cos 30° + 4)t \ \hat{\imath} + \left(-16t^2 + (15\sin 30°)t + 20\right)\hat{\jmath}$$

$$= \left(\tfrac{15}{2}\sqrt{3} + 4\right)t \ \hat{\imath} + \left(-16t^2 + \tfrac{15}{2}t + 20\right)\hat{\jmath}$$

3. Given the position vector $\vec{r}(t) = (12\sin t)\hat{\imath} + (12\cos t)\hat{\jmath} + (5t)\hat{k}$.

(4 points) a. Find the unit normal vector $\hat{N}(t)$.

$$\vec{r}'(t) = (12\cos t)\hat{\imath} + (-12\sin t)\hat{\jmath} + 5\hat{k}$$

$$|\vec{r}'(t)| = \sqrt{144\cos^2 t + 144\sin^2 t + 25} = \sqrt{144 + 25} = \sqrt{169} = 13$$

$$\hat{T}(t) = \left(\tfrac{12}{13}\cos t\right)\hat{\imath} + \left(-\tfrac{12}{13}\sin t\right)\hat{\jmath} + \tfrac{5}{13}\hat{k}$$

$$\hat{T}'(t) = \left(-\tfrac{12}{13}\sin t\right)\hat{\imath} + \left(-\tfrac{12}{13}\cos t\right)\hat{\jmath}$$

$$|\hat{T}'(t)| = \sqrt{\tfrac{144}{169}\sin^2 t + \tfrac{144}{169}\cos^2 t} = \sqrt{\tfrac{144}{169}} = \tfrac{12}{13}$$

$$\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|} = (-\sin t)\hat{\imath} + (-\cos t)\hat{\jmath}$$

(4 points) b. Find the binormal vector $\hat{B}(t)$.

$$\hat{B} = \hat{T} \times \hat{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \tfrac{12}{13}\cos t & -\tfrac{12}{13}\sin t & \tfrac{5}{13} \\ -\sin t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{\imath}\left(\tfrac{5}{13}\cos t\right) - \hat{\jmath}\left(\tfrac{5}{13}\sin t\right) + \hat{k}\left(-\tfrac{12}{13}\cos^2 t - \tfrac{12}{13}\sin^2 t\right)$$

$$= \left(\tfrac{5}{13}\cos t\right)\hat{\imath} + \left(-\tfrac{5}{13}\sin t\right)\hat{\jmath} + \left(-\tfrac{12}{13}\right)\hat{k}$$

(4 points) 4. Solve the following initial value problem for $\vec{r}(t)$:

$$\vec{v}(t) = (t^3 - 3t)\hat{i} + 4t^2\hat{j} + t\hat{k} \text{ with } \vec{r}(0) = 2\hat{i} + \hat{j}$$

$$\vec{r}(t) = \left(\frac{t^4}{4} - \frac{3}{2}t^2\right)\hat{i} + \left(\frac{4}{3}t^3\right)\hat{j} + \left(\frac{1}{2}t^2\right)\hat{k} + \vec{c}$$

$$\vec{r}(0) = 0 + \vec{c} = 2\hat{i} + \hat{j}$$

$$\vec{r}(t) = \left(\frac{t^4}{4} - \frac{3}{2}t^2 + 2\right)\hat{i} + \left(\frac{4}{3}t^3 + 1\right)\hat{j} + \left(\frac{1}{2}t^2\right)\hat{k}$$

5. Given the following function: $f(x,y) = \sqrt{49 - x^2 - y^2}$

(2 points) a. Find the function's domain.

$$49 - x^2 - y^2 \geq 0$$

$$\{(x,y) \mid x^2 + y^2 \leq 49\}$$
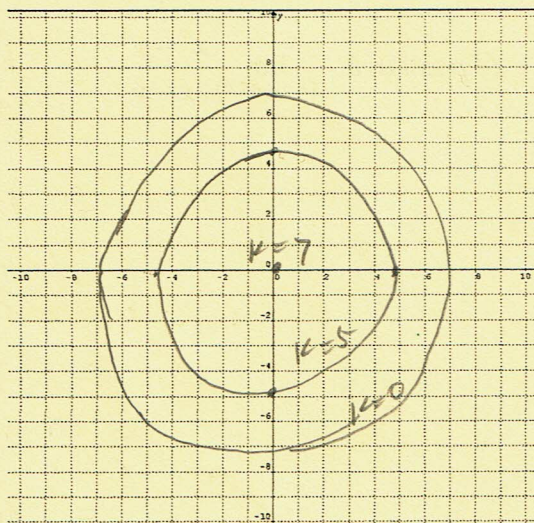
(2 points) b. Find the function's range.

$$0 \leq z \leq 7$$

(3 points) c. Sketch the function's level curves when $k = 0$, $k = 5$, and $k = 7$.

$k=0:\ \sqrt{49 - x^2 - y^2} = 0$

$49 - x^2 - y^2 = 0$

$x^2 + y^2 = 49$

$k=5\ \sqrt{49 - x^2 - y^2} = 5$

$49 - x^2 - y^2 = 25$

$x^2 + y^2 = 24$

$k=7:\ \sqrt{49 - x^2 - y^2} = 7$

$49 - x^2 - y^2 = 49$

$x^2 + y^2 = 0$

(4 points) 6. Show that $f(x,y) = \dfrac{x^6}{x^6 - y^3}$ has no limit as $(x,y) \to (0,0)$. Use $y = kx^2$, $(k \neq 1)$,

and <u>explain</u> the result.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2,\,k\neq1}} \frac{x^6}{x^6-y^3} = \lim_{\substack{(x,y)\to(0,0)\\ y=kx^2,\,k\neq1}} \frac{x^6}{x^6-(kx^2)^3} = \frac{1}{1-k^3}$$

Because the limit depends solely on k, the limit does not exist.

OR

If $k = 2$ the limit $= -\dfrac{1}{7}$

If $k = 3$ the limit $= -\dfrac{1}{26}$

Since the values of the limit are different for different values of k (via different paths), the limit does not exist.

7. Find the following limits.

(4 points) a. $\displaystyle\lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq4}} \frac{\sqrt{2x-y}-2}{2x-y-4}$

$$= \lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq4}} \frac{\sqrt{2x-y}-2}{2x-y-4} \cdot \frac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2} = \lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq4}} \frac{2x-y-4}{(2x-y-4)(\sqrt{2x-y}+2)}$$

$$= \lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq4}} \frac{1}{\sqrt{2x-y}+2} = \frac{1}{\sqrt{2(2)-0}+2} = \frac{1}{2+2} = \frac{1}{4}$$

(4 points) b. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^4-y^4}{x-y} = \lim_{(x,y)\to(2,2)} \frac{(x^2+y^2)(x^2-y^2)}{x-y}$

$$= \lim_{(x,y)\to(2,2)} \frac{(x^2+y^2)(x+y)(x-y)}{x-y}$$

$$= \lim_{(x,y)\to(2,2)} (x^2+y^2)(x+y)$$

$$= (4+4)(2+2) = (8)(4) = 32$$

8. Given $f(x,y) = y^3 e^{x^4 y^2}$. Determine the following.

(2 points) a. $f_x = 4x^3 y^5 e^{x^4 y^2}$

(2 points) b. $f_y = 3y^2 e^{x^4 y^2} + 2x^4 y^4 e^{x^4 y^2}$

(3 points) c. $f_{xx} = 12x^2 y^5 e^{x^4 y^2} + 4x^3 y^5 e^{x^4 y^2} \cdot 4x^3 y^2$

$$= 12x^2 y^5 e^{x^4 y^2} + 16x^6 y^7 e^{x^4 y^2}$$

(3 points) d. $f_{yx} = 3y^2 e^{x^4 y^2} 4x^3 y^2 + 8x^3 y^4 e^{x^4 y^2} + 2x^4 y^4 e^{x^4 y^2} 4x^3 y^2$

$$= 12x^3 y^4 e^{x^4 y^2} + 8x^3 y^4 e^{x^4 y^2} + 8x^7 y^6 e^{x^4 y^2}$$

$$= 20x^3 y^4 e^{x^4 y^2} + 8x^7 y^6 e^{x^4 y^2}$$

(3 points) e. $f_{yy} = 6y e^{x^4 y^2} + 3y^2 e^{x^4 y^2} 2x^4 y + 8x^4 y^3 e^{x^4 y^2} + 2x^4 y^4 e^{x^4 y^2} 2x^4 y$

$$= 6y e^{x^4 y^2} + 6x^4 y^3 e^{x^4 y^2} + 8x^4 y^3 e^{x^4 y^2} + 4x^8 y^5 e^{x^4 y^2}$$

$$= 6y e^{x^4 y^2} + 14x^4 y^3 e^{x^4 y^2} + 4x^8 y^5 e^{x^4 y^2}$$

(6 points) 9. Find the equations for the tangent plane and the normal line to the graph of the given equation at the point P_0.

$$x^2 + y^2 - z^2 = 18; \quad P_0(3,5,-4)$$

$\vec{\nabla} f = 2x\hat{\imath} + 2y\hat{\jmath} - 2z\hat{k}$

$\vec{\nabla} f|_{P_0} = 6\hat{\imath} + 10\hat{\jmath} + 8\hat{k}$

$6(x-3) + 10(y-5) + 8(z+4) = 0$

$6x - 18 + 10y - 50 + 8z + 32 = 0$

$6x + 10y + 8z - 36 = 0$

$3x + 5y + 4z - 18 = 0$

$x = 3 + 6t$ $x = 3 + 3t$

$y = 5 + 10t$ or $y = 5 + 5t$

$z = -4 + 8t$ $z = -4 + 4t$

(6 points) 10. Given $r = w^2 + xyz^3$, $x = 2t^2 - 7$, $y = t^4$, $z = 4t^3 + 3$. $w = 3t + 5$. Write the chain rule that is used to find $\dfrac{dr}{dt}$ and then use the chain rule to find $\dfrac{dr}{dt}$.

$$\frac{dr}{dt} = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt} + \frac{\partial r}{\partial z}\frac{dz}{dt} + \frac{\partial r}{\partial w}\frac{dw}{dt}$$

$$\frac{dr}{dt} = (yz^3)(4t) + (xz^3)(4t^3) + (3xyz^2)(12t^2) + (2w)(3)$$

$$= 4t(t^4)(4t^3+3)^3 + 4t^3(2t^2-7)(4t^3+3)^3 + 12t^2(3)(2t^2-7)(t^4)(4t^3+3)^2 + 6(3t+5)$$

$$= 4t^5(4t^3+3)^3 + 4t^3(2t^2-7)(4t^3+3)^3 + 36t^6(2t^2-7)(4t^3+3)^2 + 6(3t+5)$$

(2 points) 11. Write the chain rules that are used to find $\partial w/\partial x$ and $\partial w/\partial y$ if $w = f(r,s)$, $r = g(x,y)$ $s = h(x,y)$.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y}$$

(6 points) 12. Find the directional derivative of the following function at P_0 in the direction of $\vec{v}$:

$$f(x,y) = 2xy - 3y^2; \quad P_0(5,5); \quad \vec{v} = 4\hat{i} - 3\hat{j}$$

$\vec{\nabla}f = 2y\,\hat{\imath} + (2x - 6y)\,\hat{\jmath}$

$|\vec{v}| = \sqrt{(4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$\vec{\nabla}f\big|_p = 2(5)\,\hat{\imath} + (2(5) - 6(5))\,\hat{\jmath}$

$\hat{v} = \frac{4}{5}\hat{\imath} - \frac{3}{5}\hat{\jmath}$

$= 10\,\hat{\imath} + (10 - 30)\,\hat{\jmath}$

$= 10\,\hat{\imath} - 20\,\hat{\jmath}$

$\vec{\nabla}f\big|_p \cdot \hat{v} = (10\hat{\imath} - 20\hat{\jmath}) \cdot (\frac{4}{5}\hat{\imath} - \frac{3}{5}\hat{\jmath})$

$= \frac{40}{5} + \frac{60}{5} = \frac{100}{5} = 20$

13. Given the following: $f(x,y) = x^3 y^2 + \sqrt{y}; \quad P_0(2,1)$

(2 points) a. State the direction in which the function increases most rapidly at P_0.

$\vec{\nabla}f = 3x^2 y^2\,\hat{\imath} + (2x^3 y + \frac{1}{2}y^{-1/2})\,\hat{\jmath}$

$\vec{\nabla}f\big|_{P_0} = 3(2)^2(1)^2\,\hat{\imath} + (2(2)^3(1) + \frac{1}{2(1)^{1/2}})\,\hat{\jmath} = 12\hat{\imath} + (16 + \frac{1}{2})\hat{\jmath} = 12\hat{\imath} + \frac{33}{2}\hat{\jmath}$

33
33
99
99
1089

$|\vec{\nabla}f| = \sqrt{144 + \frac{1089}{4}} = \sqrt{\frac{576 + 1089}{4}} = \frac{\sqrt{1665}}{2}$ $\hat{u} = \frac{\vec{\nabla}f}{|\vec{\nabla}f|} = \frac{24}{\sqrt{1665}}\hat{\imath} + \frac{33}{\sqrt{1665}}\hat{\jmath}$

(2 points) b. State the direction in which the function decreases most rapidly at P_0.

$-\hat{u} = -\frac{24}{\sqrt{1665}}\hat{\imath} - \frac{33}{\sqrt{1665}}\hat{\jmath}$

(2 points) c. State the directions in which the function has zero change at P_0.

$\hat{z}_1 = \frac{33}{\sqrt{1665}}\hat{\imath} - \frac{24}{\sqrt{1665}}\hat{\jmath}$

$\hat{z}_2 = -\frac{33}{\sqrt{1665}}\hat{\imath} + \frac{24}{\sqrt{1665}}\hat{\jmath}$

(6 points) 14. Use partial derivatives to find $\dfrac{dy}{dx}$ if $e^{x^2y^2} + 6x^2y = 5x^3 + 4\sin y$, where y is a differentiable function of x.

$$F(x,y) = e^{x^2y^2} + 6x^2y - 5x^3 - 4\sin y$$

$$F_x = 2xy^2 e^{x^2y^2} + 12xy - 15x^2$$

$$F_y = 2x^2y\, e^{x^2y^2} + 6x^2 - 4\cos y$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xy^2 e^{x^2y^2} + 12xy - 15x^2}{2x^2y e^{x^2y^2} + 6x^2 - 4\cos y}$$

(6 points) 15. Find the standard linearization $L(x,y)$ of the function at the given point.

$$f(x,y) = e^{3x-y} \quad \text{at} \quad (1,3)$$

$$f_x = 3e^{3x-y} \qquad f_y = -e^{3x-y} \qquad f(3,1)=1$$

$$f_x(3,1) = 3 \qquad f_y(3,1) = -1$$

$$z - 1 = 3(x-1) - 1(y-3)$$

$$z = 1 + 3x - 3 - y + 3$$

$$z = 3x - y + 1$$

(3 points) 16. Given $V = \pi r^2 h + \dfrac{4}{3}\pi r^3$. Write the total differential dV.

$$dV = (2\pi rh + 4\pi r^2)\,dr + (\pi r^2)\,dh$$

(5 points) 17. Given $f(x,y) = xy - 2x - 2y - x^2 - y^2$. Find the critical points of f.

$$f_x = y - 2 - 2x \qquad\qquad f_y = x - 2 - 2y$$

$$y - 2 - 2x = 0 \quad \text{AND} \quad x - 2 - 2y = 0$$

$$-2x + y = 2 \qquad\qquad -2x + y = 2$$
$$2(x - 2y = 2) \qquad\qquad \underline{2x - 4y = 4}$$

$$-3y = 6 \qquad\qquad x - 2(-2) = 2$$
$$y = -2 \qquad\qquad x + 4 = 2$$
$$x = -2$$

$$(-2, -2)$$

Optional Extra Credit Question:

(4 points) 18. Use the limit definition to find the directional derivative of $f(x,y) = 4x^2 + y^2$ at the point $P_0(-1, 2)$ in the direction of $\bar{u} = 3\hat{i} - 4\hat{j}$ $|\bar{u}| = 5$ $\hat{u} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$

$$\lim_{h \to 0} \frac{f(-1 + \frac{3}{5}h, 2 - \frac{4}{5}h) - f(-1,2)}{h} = \lim_{h \to 0} \frac{\left[4(-1 + \frac{3}{5}h)^2 + (2 - \frac{4}{5}h)^2\right] - \left[4(-1)^2 + (2)^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{4\left(1 - \frac{6}{5}h + \frac{9}{25}h^2\right) + 4 - \frac{16}{5}h + \frac{16}{25}h^2 - 8}{h}$$

$$= \lim_{h \to 0} \frac{4 - \frac{24}{5}h + \frac{36}{25}h^2 + 4 - \frac{16}{5}h + \frac{16}{25}h - 8}{h}$$

$$= \lim_{h \to 0} \frac{-8h + \frac{52}{25}h^2}{h} = \lim_{h \to 0} \left(-8 + \frac{52}{25}h\right) = -8$$