MATH 280 – EXAM #2
Spring Semester 2018

Name: Key

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There are 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. Good luck!

(4 points) 1. A baseball is thrown at a speed of 25 ft/s from the stands 15 ft above the field at an angle of 60° up from the horizontal. At the instant the ball is thrown, an instantaneous horizontal gust of wind of 6 ft/s blows in the direction opposite the ball. What is the position vector that models this situation? $\left(g = 32 \text{ ft/s}^2\right)$

$\vec{r}(t) = (25\cos 60^\circ t - 6t)\,\hat{\imath} + (25\sin 60^\circ t - \frac{1}{2}(32)t^2 + 15)\,\hat{\jmath}$

$\vec{r}(t) = (\frac{25}{2}t - 6t)\,\hat{\imath} + (\frac{25\sqrt{3}}{2}t - 16t^2 + 15)\,\hat{\jmath}$

2. Given that the following position vector describes a projectile in motion where distance is in feet and time is in seconds: $\vec{r}(t) = \left(128\sqrt{3}\right)t\hat{\imath} + \left(144 + 128t - 16t^2\right)\hat{\jmath}$

(4 points) a. What is the maximum height that the projectile attains?

$\vec{r}_y{}' = 128 - 32t = 0$

$128 = 32t$

$t = 4$

$r_y(4) = 144 + 128(4) - 16(4)^2$

$= 16(9 + 8\cdot 4 - 16)$

$= 16(9 + 32 - 16)$

$= 16(9 + 16)$

$= 16\cdot 25 = 400 \text{ ft}$

(4 points) b. What is the range of the object?

$r_y = 144 + 128t - 16t^2 = 0$

$-16(t^2 - 8t - 9) = 0$

$-16(t - 9)(t + 1) = 0$

$t = 9 \quad t = -1$

9 sec

$r_x(9) = (128\sqrt{3})(9)$

$= 1152\sqrt{3} \text{ ft}$

$128 \times 9 = 1152$

3. Given the following function: $f(x,y)=\sqrt{25-x^2-y^2}$

(3 points) *a.* Find the function's domain.

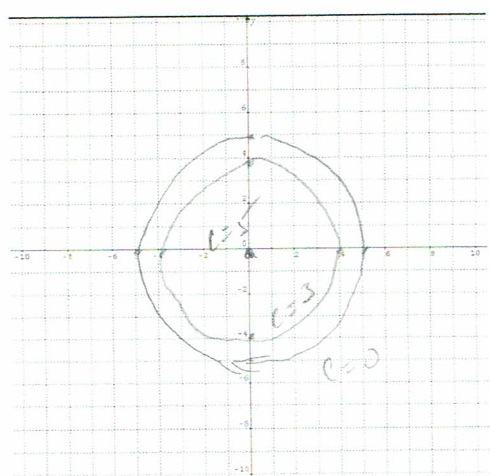
$25-x^2-y^2 \geq 0$

$25 \geq x^2+y^2$

(3 points) *b.* Find the function's range.

$0 \leq z \leq 5$

(3 points) *c.* Sketch the function's level curves when $c=0,\ c=3,\ c=5$.



$\sqrt{25-x^2-y^2}=0$

$25-x^2-y^2=0$

$25=x^2+y^2$

$\sqrt{25-x^2-y^2}=3$

$25-x^2-y^2=9$

$25-9=x^2+y^2$

$16=x^2+y^2$

$\sqrt{25-x^2-y^2}=25$

$25-x^2-y^2=25$

$0=x^2+y^2$

(4 points) 4. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+2y^2}$ does not exist using the two-path approach with $y=kx$.

$k=1$ $\quad \lim\limits_{(x,y)\to(0,0)} \dfrac{2x(x)}{x^2+2x^2} = \lim\limits_{(x,x)\to(0,0)} \dfrac{2x^2}{3x^2} = \dfrac{2}{3}$

$k=2$ $\quad \lim\limits_{(x,2x)\to(0,0)} \dfrac{2x(2x)}{x^2+2(2x)^2} = \lim\limits_{(x,2x)\to(0,0)} \dfrac{4x^2}{x^2+8x^2} = \dfrac{4}{9}$

5. Find the following limits.

(4 points) *a.* $\lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \dfrac{x+y-4}{\sqrt{x+y}-2} \cdot \dfrac{\sqrt{x+y}+2}{\sqrt{x+y}+2}$

$= \lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \dfrac{(x+y-4)(\sqrt{x+y}+2)}{x+y-4}$

$= \lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} (\sqrt{x+y}+2)) = \sqrt{2+2}+2 = \sqrt{4}+2$
$= 2+2 = 4$

(4 points) *b.* $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{xy-y-2x+2}{x-1}$ $= \lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{y(x-1)-2(x-1)}{x-1}$

$= \lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} (y-2) = 1-2 = -1$

(6 points) 6. Find the equations for the tangent plane and the normal line to the graph of the given equation at the point P_0.

$$x^3-2xy+z^3+7y+6=0; \quad P_0(1,4,-3)$$

$\vec{\nabla} f = (3x^2-2y)\hat{i} + (-2x+7)\hat{j} + (3z^2)\hat{k}$

$\vec{\nabla} f\big|_{P_0} = (3(1)^2-2(4))\hat{i} + (-2(1)+7)\hat{j} + (3(-3)^2)\hat{k}$

$= (3-8)\hat{i} + (-2+7)\hat{j} + (3 \cdot 9)\hat{k}$

$= -5\hat{i} + 5\hat{j} + 27\hat{k}$

Tangent Line

81
-15
66

$-5(x-1)+5(y-4)+27(z+3)=0$

$-5x+5+5y-20+27z+81=0$

$-5x+5y+27z+66=0$

Normal Line

$x = 1+5t$

$y = 4+5t$

$z = -3+27t$

7. Given $f(x,y)=e^{x^2y^3}$. Determine the following.

(3 points) a. f_x $= e^{x^2y^3} \cdot 2xy^3$

(3 points) b. f_y $= e^{x^2y^3}\, 3x^2y^2$

(3 points) c. f_{xx} $= e^{x^2y^3}\, 2xy^3 \cdot 2xy^3 + e^{x^2y^3} \cdot 2y^3$

$= e^{x^2y^3}\, 4x^2y^6 + e^{x^2y^3}\, 2y^3$

$= e^{x^2y^3}\left(4x^2y^6 + 2y^3\right)$

(3 points) d. f_{yx} $= e^{x^2y^3}\, 2xy^3\, 3x^2y^2 + e^{x^2y^3}\, 6xy^2$

$= e^{x^2y^3}\, 6x^3y^5 + e^{x^2y^3}\, 6xy^2$

$= e^{x^2y^3}\left(6x^3y^5 + 6xy^2\right)$

(3 points) e. f_{yy} $= e^{x^2y^3}\, 3x^2y^2\, 3x^2y^2 + e^{x^2y^3}\, 6x^2y$

$= e^{x^2y^3}\, 9x^4y^4 + e^{x^2y^3}\, 6x^2y$

$= e^{x^2y^3}\left(9x^4y^4 + 6x^2y\right)$

(6 points) *9.* Given $w = x^2 + yz$, $x = 3t^2 + 1$, $y = 2t - 4$, $z = t^3$. Write the chain rule that is used to find $\dfrac{dw}{dt}$ and then use the chain rule to find $\dfrac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= 2x \cdot 6t + z \cdot 2 + y \cdot 3t^2$$

$$= 2(3t^2+1)6t + t^3 \cdot 2 + (2t-4)3t^2$$

$$= 36t^3 + 12t + 2t^3 + 6t^3 - 12t^2$$

$$= 44t^3 - 12t^2 + 12t$$

(2 points) *10.* Write the chain rules that are used to find $\partial w/\partial x$ and $\partial w/\partial y$ if $w = f(r,s,t,v)$, $r = g(x,y)$ $s = h(x,y)$, $t = k(x,y)$, $v = l(x,y)$.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

11. Given the following: $f(x,y) = x^2y + e^{xy}\sin y$; $P_0(1,0)$

(3 points) *a.* State the directions in which the function increases and decreases most rapidly at P_0.

$$\vec{\nabla} f = (2xy + ye^{xy}\sin y)\hat{i} + (x^2 + xe^{xy}\sin y + e^{xy}\cos y)\hat{j}$$

$$\vec{\nabla} f|_{P_0} = (2\cdot1\cdot0 + 1e^{1\cdot0}\sin 0)\hat{i} + (1^2 + 1\cdot e^{1\cdot0}\sin 0 + e^{1\cdot0}\cos 0)\hat{j}$$

$$= 0\hat{i} + (1+1)\hat{j} = 2\hat{j}$$

most rapid increase $= \hat{j}$ most rapid decrease $= -\hat{j}$

(3 points) *b.* State the directions in which the function has zero change at P_0.

$\hat{i}$, $-\hat{i}$

(6 points) *11.* Use partial derivatives to find $\frac{dy}{dx}$ if $\cos(xy) = x^2y^4 + x\sin y$, where y is a differentiable function of x.

$F(x,y) = \cos(xy) - x^2y^4 - x\sin y = 0$

$F_x = -\sin(xy)y - 2xy^4 - \sin y$

$F_y = -\sin(xy)x - 4x^2y^3 - x\cos y$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\sin(xy) - 2xy^4 - \sin y}{-x\sin(xy) - 4x^2y^3 - x\cos y} = -\frac{y\sin(xy) + 2xy^4 + \sin y}{x\sin(xy) + 4x^2y^3 + x\cos y}$$

(6 points) *12.* Find the standard linearization $L(x,y)$ of the function at the given point.

$$f(x,y) = \sqrt{x + e^{4y}} \text{ at } (3,0)$$

$f(3,0) = \sqrt{3 + e^{4(0)}} = \sqrt{3+1} = 2$

$f_x = \frac{1}{2}(x + e^{4y})^{-1/2} \cdot 1$ $\quad f_y = \frac{1}{2}(x + e^{4y})^{-1/2} e^{4y} \cdot 4$

$f_x\big|_{(3,0)} = \frac{1}{2\sqrt{3 + e^{4(0)}}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ $\quad f_y\big|_{(3,0)} = \frac{4e^{4(0)}}{2\sqrt{3 + e^{4(0)}}} = \frac{4}{2\sqrt{4}} = 1$

$L(x,y) = 2 + \frac{1}{4}(x-3) + 1(y-0) = 2 + \frac{1}{4}x - \frac{3}{4} + y$

$= \frac{1}{4}x + y + \frac{5}{4}$

(6 points) *13.* Find the directional derivative of the following function at P_0 in the direction of $\vec{v}$:

$$f(x,y,z) = xy^3z^2; \quad P_0(2,-1,4); \quad \vec{v} = \hat{i} + 2\hat{j} - 3\hat{k}$$

$|\vec{v}| = \sqrt{1+4+9} = \sqrt{14}$

$\vec{\nabla}f = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$ $\quad \hat{v} = \frac{\langle 1,2,-3\rangle}{\sqrt{14}}$

$\vec{\nabla}f\big|_{P_0} = (-1)^3(4)^2\hat{i} + 3(2)(-1)^2(4)^2\hat{j} + 2(2)(-1)^3(4)\hat{k}$

$= -16\hat{i} + 96\hat{j} - 16\hat{k}$

$\vec{\nabla}f \cdot \hat{v} = \langle -16, 96, -16\rangle \cdot \frac{\langle 1,2,-3\rangle}{\sqrt{14}} = \frac{-16 + 192 + 48}{\sqrt{14}} = \frac{224}{\sqrt{14}}$

(8 points) *14.* Given $f(x,y) = x^2 - 4xy + y^3 + 4y$. Find all local maxima, local minima, and saddle points.

$f_x = 2x - 4y$

$2x - 4y = 0$

$2x = 4y$

$f_y = -4x + 3y^2 + 4$

$-4x + 3y^2 + 4 = 0$

$-8y + 3y^2 + 4 = 0$

$3y^2 - 8y + 4 = 0$

$(3y-2)(y-2) = 0$

$y = 2/3 \quad y = 2$

$2x = 4(\frac{2}{3}) \quad 2x = 4(2)$

$x = \frac{4}{3} \quad 2x = 8$

$x = 4$

$(\frac{4}{3}, \frac{2}{3}) \quad (4, 2)$

$f_{xx} = 2 \quad f_{xy} = -4 \quad f_{yy} = 6y$

$f_{xx} f_{yy} - (f_{xy})^2$

$12y - 16$

$(\frac{4}{3}, \frac{2}{3})$: $12(\frac{2}{3}) - 16 = 8 - 16 < 0$

saddle point

$(4, 2)$ $12(2) - 16 = 24 - 16 > 0$

$2 > 0$ local minimum

(8 points) *15.* Use Lagrange Multipliers to find the largest product the positive numbers *x*, *y*, and *z* can have if $x + y + z^2 = 16$

$f(x,y,z) = xyz \qquad g(x,y,z) = x + y + z^2 = 16$

$\vec{\nabla} f = yz\hat{i} + xz\hat{j} + xy\hat{k} \qquad \vec{\nabla} g = \hat{i} + \hat{j} + 2z\hat{k} \qquad \vec{\nabla} f = \lambda \vec{\nabla} g$

$yz = \lambda \quad xz = \lambda \quad xy = 2z\lambda$ and $x + y + z^2 = 16$

$yz = xz \qquad xy = 2z(xz)$

$z \neq 0 \quad y = x \qquad xy = 2xz^2$

$x \neq 0 \quad y = 2z^2$

$\frac{y}{2} = z^2$

$y + y + \frac{y}{2} = 16$

$\frac{5}{2} y = 16$

$y = \frac{32}{5}$

$x = \frac{32}{5}$

$z^2 = \frac{16}{5}$

$z = \frac{4}{\sqrt{5}}$

$x = \frac{32}{5}, y = \frac{32}{5}, z = \frac{4}{\sqrt{5}}$ product $= \frac{4096}{25\sqrt{5}}$