MATH 280 SPRING 2022 Exam #2 Key

(4pts)

#1.) $\vec{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$

$\vec{r}\,'(t) = \langle 2, 2t, t^2 \rangle$

$$\int_0^1 \sqrt{4+4t^2+t^4}\,dt = \int_0^1 \sqrt{(t^2+2)^2}\,dt = \int_0^1 (t^2+2)\,dt = \frac{t^3}{3} + 2t \Big|_0^1$$

$$= \frac{1}{3} + 2 = \frac{7}{3}$$

(4pts)

#2.) $$K = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \hat{i}(12t^2 - 6t^2) - \hat{j}(6t) + \hat{k}(2)$$

$$= 6t^2\hat{i} - 6t\hat{j} + 2\hat{k}$$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{36t^4+36t^2+4} = \sqrt{4(9t^2+9t+1)} = 2\sqrt{9t^2+9t+1}$$

$$|\vec{r}\,'(t)| = \sqrt{1+4t^2+9t^4}$$

$$K = \frac{2\sqrt{9t^2+9t+1}}{(1+4t^2+9t^4)^{3/2}}$$

$t=1$ $$K = \frac{2\sqrt{19}}{14^{3/2}}$$

#3.) $\vec{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$ $\quad P_0(1,0,0) \to t=0$

(4pts)
a.) $\vec{r}\,'(t) = \langle -\sin t, \cos t, -\frac{\sin t}{\cos t} \rangle = \langle -\sin t, \cos t, -\tan t \rangle$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1+\tan^2 t} = \sqrt{\sec^2 t} = \sec t$$

$$\hat{T}(t) = \frac{\langle -\sin t, \cos t, -\tan t \rangle}{\sec t} = \langle -\sin t\cos t, \cos^2 t, -\sin t \rangle$$

$$= \langle -\tfrac{1}{2}\sin 2t, \cos^2 t, -\sin t \rangle \qquad \hat{T}(0) = \langle 0, 1, 0 \rangle$$

$$\hat{T}\,'(t) = \langle -\cos 2t, -2\cos t\sin t, -\cos t \rangle = \langle -\cos 2t, -\sin 2t, -\cos t \rangle$$

$$|T'(t)| = \sqrt{\cos^2 2t + \sin^2 2t + \cos^2 t} = \sqrt{1+\cos^2 t}$$

$$\hat{N}(t) = \left\langle \frac{-\cos 2t}{\sqrt{1+\cos^2 t}}, \frac{-\sin 2t}{\sqrt{1+\cos^2 t}}, \frac{-\cos t}{\sqrt{1+\cos^2 t}} \right\rangle \qquad \hat{N}(0) = \langle \tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}} \rangle$$

(4pts)
b.)

$$\hat{B} = \hat{T} \times \hat{N}$$

$$\hat{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} = \hat{i}\left(-\tfrac{1}{\sqrt{2}}\right) - \hat{j}(0) + \hat{k}\left(-\tfrac{1}{\sqrt{2}}\right)$$

$$= \langle -\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}} \rangle$$

(3pts)
#4) $\vec{a}(t) = (4t+2)\hat{i} + (5t)\hat{j} + (3t^2-1)\hat{k}$

$$\vec{v}(t) = (2t^2+2t)\hat{i} + (\tfrac{5}{2}t^2)\hat{j} + (t^3-t)\hat{k} + \vec{C}_1$$

$$\vec{v}(0) = 4\hat{i} + 3\hat{j} - 2\hat{k} = \vec{C}_1$$

$$\vec{v}(t) = (2t^2+2t+4)\hat{i} + (\tfrac{5}{2}t^2+3)\hat{j} + (t^3-t-2)\hat{k}$$

$$\vec{r}(t) = (\tfrac{2}{3}t^3+t^2+4t)\hat{i} + (\tfrac{5}{6}t^3+3t)\hat{j} + (\tfrac{1}{4}t^4-\tfrac{1}{2}t^2-2t)\hat{k} + \vec{C}_2$$

$$\vec{r}(0) = 3\hat{i} + 5\hat{k} = \vec{C}_2$$

$$\vec{r}(t) = (\tfrac{2}{3}t^3+t^2+4t+3)\hat{i} + (\tfrac{5}{6}t^3+3t)\hat{j} + (\tfrac{1}{4}t^4-\tfrac{1}{2}t^2-2t+5)\hat{k}$$

#5.) $\vec{r}(t) = (80\sqrt{3})t\hat{i} + (96 + 80t - 16t^2)\hat{j}$

(3 pts)
a.) $r_y = 96 + 80t - 16t^2$

$r_y' = 80 - 32t = 0$

$80 = 32t$

$t = 2.5$ sec.

(2 pts)
b.) $r_y(2.5) = 96 + 80(2.5) - 16(2.5)^2$

$= 196$ ft

(3 pts)
c.) $96 + 80t - 16t^2 = 0$

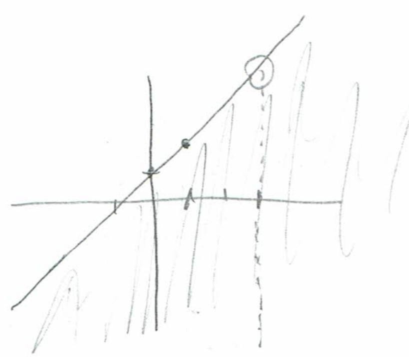
$-16(t^2 - 5t - 6) = 0$

$-16(t-6)(t+1) = 0$

$t = 6$ sec.

(2 pts)
d.) $r_x(6) = 80\sqrt{3}(6) = 480\sqrt{3}$ ft

(4 pts)
#6.) $\{(x,y) \mid x \neq 3,\ x - y + 1 \geq 0\}$



$x + 1 \geq y$

$(0,0)$: $1 \geq 0$ T

(4 pts)
#7) $f(x,y) = x^2 + y$

$k=0$ $\quad x^2 + y = 0$

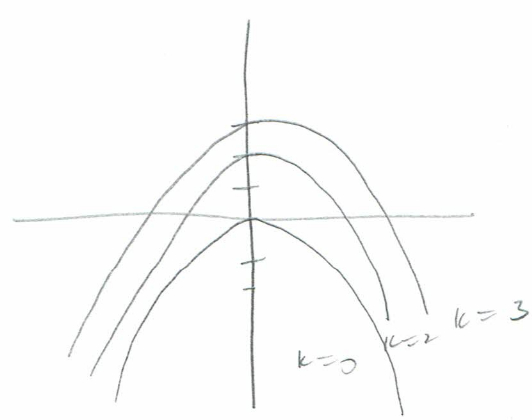
$y = -x^2$

$k=3$ $\quad x^2 + y = 3$

$y = -x^2 + 3$

$k=2$ $\quad x^2 + y = 2$

$y = -x^2 + 2$



(4pts)
#8.)

$$\lim_{(x,y)\to(0,0)} \frac{x^4-16y^4}{x^2-4y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+4y^2)(x^2-4y^2)}{x^2-4y^2} = \lim_{(x,y)\to(0,0)} (x^2+4y^2) = 0$$

(4pts)
#9.)

$$\lim_{(x,y)\to(0,0)} \frac{2x^2-y^2}{x^2+y^2}$$

x-axis: $\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{2x^2-y^2}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{2x^2}{x^2} = 2$

y-axis: $\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{2x^2-y^2}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{-y^2}{y^2} = -1$

$\therefore$ By the two-paths approach, the limit DNE.

#10.) $f(x,y) = y^2 \sin(x \cdot y)$

(2pts)
a.) $f_x = y^3 \cos(xy)$

(2pts)
b.) $f_y = 2y\sin(xy) + xy^2\cos(xy)$

(3pts)
c.) $f_{xx} = -y^4 \sin(xy)$

(3pts)
d.) $f_{xy} = 3y^2\cos(xy) - xy^3\sin(xy)$

(3pts)
e.) $f_{yy} = 2\sin(xy) + 2xy\cos(xy) + 2xy\cos(xy) - x^2y^2\sin(xy)$

$= 2\sin(xy) + 4xy\cos(xy) - x^2y^2\sin(xy)$

(4pts)
11)

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$=(8xy^3)(15s^2) + (12x^2y^2)(32s^3) + (8z^3)(15s^4)$$

$$= 120xy^3s^2 + 384x^2y^2s^3 + 120z^3s^4$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$=(8xy^3)(14t) + (12x^2y^2)(21t^2) + (8z^3)(-24t^3)$$

$$= 112xy^3t + 252x^2y^2t^2 - 192z^3t^3$$

(4pts)
#12) $\frac{dy}{dx} = -\frac{F_x}{F_y}$ $\quad F(x,y) = 4x^3 + 6x^4y^5 - 3\sin(x^2y^3) - 5y^4 = 0$

$$F_x = 12x^2 + 24x^3y^5 - 6xy^3\cos(x^2y^3)$$

$$F_y = 30x^4y^4 - 9x^2y^2\cos(x^2y^3) - 20y^3$$

$$\frac{dy}{dx} = -\frac{12x^2 + 24x^3y^5 - 6xy^3\cos(x^2y^3)}{30x^4y^4 - 9x^2y^2\cos(x^2y^3) - 20y^3}$$

#13) $f(x,y) = x^3e^{xy}$ $\quad f(1,0) = 1$ $\quad f_x = 3x^2e^{xy} + x^3ye^{xy}$ $\quad f_y = x^4e^{xy}$

(4pts)
a) $z - z_0 = f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$z - 1 = 3(x-1) + 1(y-0)$

$z - 1 = 3x - 3 + y$ $\quad z = 3x + y - 2 = L(x,y)$

(2pts)
b) $L(1.3, 0.2) = z = 3(1.3) + 0.2 - 2 = 2.1$

(4 pts)
#14) $f(x,y,z) = \cos(xy) + e^{yz} + \ln(zx)$

$\vec{u} = \hat{i} + 2\hat{j} + 2\hat{k}$

$|\vec{u}| = \sqrt{1+4+4} = 3$

$\hat{u} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

$\vec{\nabla} f = \left(-y\sin(xy) + \frac{z}{zx}\right)\hat{i} + \left(-x\sin(xy) + ze^{yz}\right)\hat{j}$

$+ \left(ye^{yz} + \frac{x}{zx}\right)\hat{k}$

$= \left(-y\sin(xy) + \frac{1}{x}\right)\hat{i} + \left(-x\sin(xy) + ze^{yz}\right)\hat{j} + \left(ye^{yz} + \frac{1}{z}\right)\hat{k}$

$\vec{\nabla} f(1,0,\frac{1}{2}) = 1\hat{i} + \frac{1}{2}\hat{j} + 2\hat{k}$

$D_{\hat{u}}(1,0,\frac{1}{2}) = \vec{\nabla} f \cdot \hat{u} = \left(1\hat{i} + \frac{1}{2}\hat{j} + 2\hat{k}\right) \cdot \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

$= \frac{1}{3} + \frac{1}{3} + \frac{4}{3} = \frac{6}{3} = 2$

#15.) $x^2 + 2xy - y^2 + z^2 = 7$

(4 pts)
a.) $f(x,y,z) = x^2 + 2xy - y^2 + z^2 = 7$

$\vec{\nabla} f = (2x+2y)\hat{i} + (2x-2y)\hat{j} + (2z)\hat{k}$

$\vec{\nabla} f(1,-1,3) = 0\hat{i} + 4\hat{j} + 6\hat{k}$

$0(x-1) + 4(y+1) + 6(z-3) = 0$

$4y + 4 + 6z - 18 = 0$

$4y + 6z - 14 = 0$

(2 pts)
b) $x = 1$

$y = -1 + 4t$

$z = 3 + 6t$

#16) $f(x,y) = ye^{xy}$ $P: (0, 2)$

(2pts)

a) $\vec{\nabla} f = (y^2 e^{xy})\hat{i} + (e^{xy} + xe^{xy})\hat{j}$

$\vec{\nabla} f(0,2) = 4\hat{i} + 1\hat{j}$

(2pts)

b) $-\vec{\nabla} f = -4\hat{i} - 1\hat{j}$

(2pts)

c) $\vec{z}_1 = \hat{i} - 4\hat{j}$

$\vec{z}_2 = -\hat{i} + 4\hat{j}$