MATH 280 – EXAM #2 **Name:** KEY

Summer Session 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There are 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. Good luck!

1. Given that the following position vector describes a projectile in motion where distance is in feet and time is in seconds: $\vec{r}(t)=\left(64\sqrt{3}\right)t\hat{i}+\left(80+64t-16t^2\right)\hat{j}$

(3 points) a. What is the maximum height that the projectile attains?

$r_y = 80 + 64t - 16t^2$

$r_y' = 64 - 32t = 0$

$64 = 32t$

$t = 2$

$r_y(2) = 80 + 64(2) - 16(2)^2$

$= 80 + 128 - 64$

$= 144$ ft

(3 points) b. What is the range of the object?

$r_y = 80 + 64t - 16t^2 = 0$

$-16(t^2 - 4t - 5) = 0$

$-16(t-5)(t+1) = 0$

$t = 5, t = -1$

$t = 5$

$r_x(5) = 64\sqrt{3}(5)$

$= 320\sqrt{3}$ ft.

(3 points) 2. A baseball is thrown at a speed of 20 ft/s from the stands 10 ft above the field at an angle of 60° up from the horizontal. What is the position vector that models this situation? $\left(g = 32 \text{ ft/s}^2\right)$

$\vec{r}(t) = (20\cos 60°)t\hat{\imath} + (-\frac{1}{2}\cdot 32t^2 + 20(\sin 60°)t + 10)\hat{\jmath}$

$\vec{r}(t) = \cancel{10}\ 10t\ \hat{\imath} + (-16t^2 + 10\sqrt{3}t + 10)\hat{\jmath}$

3. Given the position vector $\vec{r}(t) = (4\cos t)\hat{i} + (3t)\hat{j} + (4\sin t)\hat{k}$.

(4 points) a. Find the unit normal vector $\hat{N}(t)$.

$\vec{r}\,'(t) = (-4\sin t) + 3\hat{j} + (4\cos t)\hat{k}$

$|\vec{r}\,'(t)| = \sqrt{16\sin^2 t + 9 + 16\cos^2 t} = 5$

$\hat{T}(t) = -\frac{4}{5}\sin t\,\hat{i} + \frac{3}{5}\hat{j} + \frac{4}{5}\cos t\,\hat{k}$

$\hat{T}'(t) = -\frac{4}{5}\cos t\,\hat{i} + 0\hat{j} - \frac{4}{5}\sin t\,\hat{k}$

$|\hat{T}'(t)| = \sqrt{\frac{16}{25}\cos^2 t + \frac{16}{25}\sin^2 t} = \frac{4}{5}$

$\hat{N}(t) = -\cos t\,\hat{i} - \sin t\,\hat{k}$

(4 points) b. Find the binormal vector $\hat{B}(t)$.

$$\hat{B} = \hat{T} \times \hat{N} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\frac{4}{5}\sin t & \frac{3}{5} & \frac{4}{5}\cos t \\ -\cos t & 0 & -\sin t \end{vmatrix} = \hat{i}\left(-\frac{3}{5}\sin t\right) - \hat{j}\left(\frac{4}{5}\sin^2 t + \frac{4}{5}\cos^2 t\right) + \hat{k}\left(\frac{3}{5}\cos t\right)$$

$$= -\frac{3}{5}\sin t\,\hat{i} - \frac{4}{5}\hat{j} + \frac{3}{5}\cos t\,\hat{k}$$

(4 points) c. Find the curvature κ.

$$\kappa = \frac{|\hat{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{4/5}{5} = 4/25$$

4. Given the following function: $f(x,y)=\sqrt{16-x^2-y^2}$

(2 points) *a.* Find the function's domain.

$16-x^2-y^2 \geq 0$

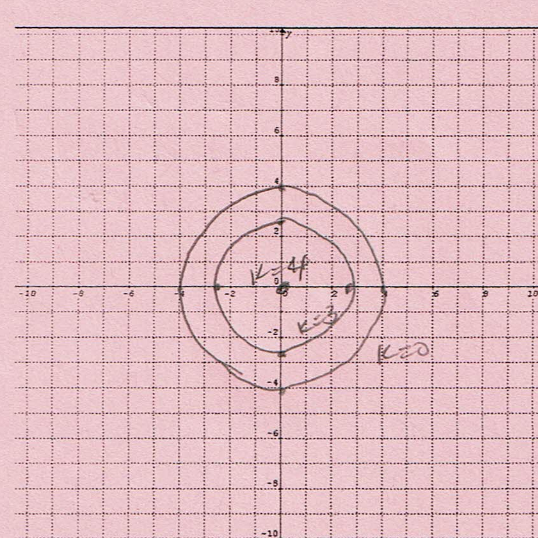
$x^2+y^2 \leq 16$

$\{(x,y) \mid x^2+y^2 \leq 16\}$

(2 points) *b.* Find the function's range.

$\{z \mid 0 \leq z \leq 4\}$

(3 points) *c.* Sketch the function's level curves when $k=0$, $k=3$, and $k=4$.



$k=0$: $\sqrt{16-x^2-y^2}=0$

$16-x^2-y^2=0$

$x^2+y=16$

$k=3$ $\sqrt{16-x^2-y^2}=3$

$16-x^2-y^2=9$

$x^2+y^2=7$

$k=4$: $\sqrt{16-x^2-y^2}=4$

$16-x^2-y^2=16$

$x^2+y^2=0$

(4 points) 5. Show that $f(x,y)=\dfrac{x^4}{x^4-y^2}$ has no limit as $(x,y)\to(0,0)$. Use $y=kx^2$, $(k\neq 1)$, and <u>explain</u> the result.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2,\,k\neq1}} \frac{x^4}{x^4-y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=kx^2,\,k\neq1}} \frac{x^4}{x^4-(kx^2)^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=kx^2,\,k\neq}} \frac{x^4}{x^4-k^2x^4} = \lim_{\substack{(x,y)\to(0,0)\\ y=kx^2,\,k\neq1}} \frac{1}{1-k^2} = \frac{1}{1-k^2}$$

The limit depends solely on k, so it will depend on the path of $y=kx^2$.

Thus, if $k=0$, the limit equals 1 and if $k=2$ the limit equals. $-1/3$.

So, the limit is dependent of path and will give different limits.

Therefore, the limit does not exist.

6. Find the following limits.

$\to \frac{0}{0}$

(4 points) a. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{xy-y-3x+3}{x-1}$

$= \lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{y(x-1)-3(x-1)}{x-1} = \lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{(y-3)(x-1)}{x-1} = \lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} (y-3) = -2$

$\to \frac{0}{0}$

(4 points) b. $\lim\limits_{\substack{(x,y)\to(5,4)\\ x+y\neq 9}} \dfrac{x+y-9}{\sqrt{x+y}-3} \cdot \dfrac{\sqrt{x+y}+3}{\sqrt{x+y}+3} = \lim\limits_{\substack{(x,y)\to(5,4)\\ x+y\neq 9}} \dfrac{(x+y-9)(\sqrt{x+y}+3)}{x+y-9}$

$= \lim\limits_{\substack{(x,y)\to(5,4)\\ x+y\neq 9}} (\sqrt{x+y}+3) = \sqrt{5+4}+3 = 6$

(6 points) 7. Find the equations for the tangent plane and the normal line to the graph of the given equation at the point P_0.

$$x^2+2xy-y^2+z^2=7; \quad P_0(1,-1,3)$$

$\vec{\nabla} f = (2x+2y)\hat{i} + (2x-2y)\hat{j} + 2z\hat{k}$

$\vec{\nabla} f|_{P_0} = 0\hat{i} + 4\hat{j} + 6\hat{k}$

$0(x-1) + 4(y+1) + 6(z-3) = 0$

$4y+4+6z-18=0$

$4y+6z=14$

$2y+3z=7$

$x = 1 +$

$y = -1+4t$

$z = 3+6t$

8. Given $f(x,y)=\sin\left(x^2y^3\right)$. Determine the following.

(2 points) a. $f_x = 2xy^3\cos(x^2y^3)$

(2 points) b. $f_y = 3x^2y^2\cos(x^2y^3)$

(3 points) c. $f_{xx} = 2y^3\cos(x^2y^3) - 4x^2y^6\sin(x^2y^3)$

(3 points) d. $f_{yx} = 6xy^2\cos(x^2y^3) - 6x^3y^5\sin(x^2y^3)$

(3 points) e. $f_{yy} = 6x^2y\cos(x^2y^3) - 9x^4y^4\sin(x^2y^3)$

(6 points) *9.* Given $w = y^2 + xz$, $x = 2t^2 + 3$, $y = t^3$, $z = 3t + 4$. Write the chain rule that is used to find $\frac{dw}{dt}$ and then use the chain rule to find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= z(4t) + 2y(3t^2) + x(3)$$

$$= (3t+4)(4t) + 2t^3(3t^2) + (2t^2+3)3$$

$$= 12t^2 + 16t + 6t^5 + 6t^2 + 9$$

$$= 6t^5 + 18t^2 + 16t + 9$$

(2 points) *10.* Write the chain rules that are used to find $\partial w/\partial x$ and $\partial w/\partial y$ if $w = f(r,s,t)$, $r = g(x,y)$ $s = h(x,y)$, $t = k(x,y)$.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$$

(6 points) *11.* Find the directional derivative of the following function at P_0 in the direction of $\vec{v}$:

$$f(x,y,z) = x^2y + x\sqrt{1+z}; \quad P_0(1,2,3); \quad \vec{v} = 2\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{\nabla} f = (2xy + \sqrt{1+z})\hat{\imath} + (x^2)\hat{\jmath} + x\tfrac{1}{2}(1+z)^{-1/2}\hat{k}$$

$$\vec{\nabla} f\big|_{P_0} = (4 + \sqrt{1+3})\hat{\imath} + 1\hat{\jmath} + \frac{1}{2\sqrt{1+3}}\hat{k}$$

$$= 6\hat{\imath} + \hat{\jmath} + \tfrac{1}{4}\hat{k}$$

$$|\vec{v}| = \sqrt{4+1+4} = 3 \qquad \hat{v} = \tfrac{2}{3}\hat{\imath} + \tfrac{1}{3}\hat{\jmath} - \tfrac{2}{3}\hat{k}$$

$$D_{\hat{u}} f = \vec{\nabla} f\big|_{P_0} \cdot \hat{v} = (6\hat{\imath} + \hat{\jmath} + \tfrac{1}{4}\hat{k}) \cdot (\tfrac{2}{3}\hat{\imath} + \tfrac{1}{3}\hat{\jmath} - \tfrac{2}{3}\hat{k})$$

$$= 4 + \tfrac{1}{3} - \tfrac{1}{6} = \frac{24+2-1}{6} = \frac{25}{6}$$

12. Given the following: $f(x,y) = ye^{xy}$; $P_0(0,2)$

(3 points) *a.* State the direction in which the function increases most rapidly at P_0.

$\vec{\nabla} f = y^2 e^{xy}\,\hat{\imath} + (e^{xy} + xye^{xy})\,\hat{\jmath}$

$= 4\hat{\imath} + \hat{\jmath}$

$|\vec{\nabla} f| = \sqrt{16+1} = \sqrt{17}$ $\qquad \hat{u} = \frac{4}{\sqrt{17}}\hat{\imath} + \frac{1}{\sqrt{17}}\hat{\jmath}$

(3 points) *b.* State the direction in which the function decreases most rapidly at P_0.

$\hat{u} = -\frac{4}{\sqrt{17}}\hat{\imath} - \frac{1}{\sqrt{17}}\hat{\jmath}$

(3 points) *c.* State the directions in which the function has zero change at P_0.

$\hat{u}_1 = \frac{1}{\sqrt{17}}\hat{\imath} - \frac{4}{\sqrt{17}}\hat{\jmath}$

$\hat{u}_2 = -\frac{1}{\sqrt{17}}\hat{\imath} + \frac{4}{\sqrt{17}}\hat{\jmath}$

(6 points) *13.* Use partial derivatives to find $\frac{dy}{dx}$ if $\sin(x^2y) = x^3y^2 + x^2\cos y$, where y is a differentiable function of x.

$f(x,y) = \sin(x^2y) - x^3y^2 - x^2\cos y = 0$

$F_x = 2xy\cos(x^2y) - 3x^2y^2 - 2x\cos y$

$F_y = x^2\cos(x^2y) - 2x^3y + x^2\sin y$

$\frac{dy}{dx} = -\frac{F_x}{F_y}$

$\frac{dy}{dx} = -\frac{2xy\cos(x^2y) - 3x^2y^2 - 2x\cos y}{x^2\cos(x^2y) - 2x^3y + x^2\sin y}$

(6 points) *14.* Find the standard linearization $L(x,y)$ of the function at the given point.

$$f(x,y)=e^x\cos y \text{ at } (0,\pi/2)$$

$f_x = e^x \cos y$ $f_y = -e^x \sin y$ $f(0,\pi/2)=0$

$f_x(0,\pi/2)=0$ $f_y(0,\pi/2)=-1$

$L(x,y) = f(0,\pi/2) + f_x(0,\pi/2)(x-0) + f_y(0,\pi/2)(y-\pi/2)$

$L(x,y) = 0 + 0(x-0) - 1(y-\pi/2)$

$L(x,y) = -y + \pi/2$

(8 points) *15.* Given $f(x,y)=x^2-2x^3+3y^2+6xy$. Find all local maxima, local minima, and saddle points.

$f_x = 2x - 6x^2 + 6y$ $f_y = 6y + 6x$

$2x - 6x^2 + 6y = 0$ $6y + 6x = 0$

$y = -x$

$2x - 6x^2 - 6x = 0$

$-6x^2 - 4x = 0$

$-2x(3x+2) = 0$

$x = 0, -2/3$ $x=0$ $y=0$

$x=-2/3$ $y=2/3$

$f_{xx} = 2 - 12x$ $f_{yy} = 6$ $f_{xy} = 6$

$(0,0)$: $D = (2)(6) - (6)^2 = 12 - 36 = -24 < 0$ $(0,0)$ saddle point

$(-\frac{2}{3}, \frac{2}{3})$ $D = (10)(6) - (6)^2 = 60 - 36 = 24 > 0$

$f_{xx} = 10 > 0$ $(-\frac{2}{3}, \frac{2}{3})$ local minimum.