MATH 280 – EXAM #2 **Name:** Key

Fall Semester 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There are 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. Good luck!

1. Given that the following position vector describes a projectile in motion where distance is in feet and time is in seconds: $\bar{r}(t) = \left(48\sqrt{3}\right)t\hat{i} + \left(160 + 48t - 16t^2\right)\hat{j}$

(3 points) *a.* What is the maximum height that the projectile attains?

$r_y' = 48 - 32t$

$48 - 32t = 0$

$48 = 32t$

$\frac{48}{32} = t$

$t = \frac{3}{2} = 1.5 \text{ sec.}$

$r_y = 160 + 48(1.5) - 16(1.5)^2$

$= 160 + 72 - 16(2.25)$

$= 160 + 72 - 36$

$= 196 \text{ ft}$

2.25×16: 1350, 225, 36.00

(3 points) *b.* When does the projectile hit the ground?

$160 + 48t - 16t^2 = 0$

$-16(t^2 - 3t - 10) = 0$

$-16(t-5)(t+2) = 0$

$t = 5 \text{ sec.}$

(3 points) *c.* What is the range of the projectile?

$r_x(5) = (48\sqrt{3})(5) = 240\sqrt{3} \text{ ft}$

(3 points) *2.* A baseball is thrown at a speed of 20 ft/s from the stands 35 ft above the field at an angle of 30° up from the horizontal. At the instant the ball is thrown, an instantaneous horizontal gust of wind of 6 ft/s blows against the direction in which the ball is thrown. What is the position vector that models this situation? $\left(g = 32 \text{ ft/s}^2\right)$

$$r(t) = (10\sqrt{3} - 6)t\,\hat{i} + (-16t^2 + 10t + 35)\,\hat{j}$$

3. Given the position vector $\vec{r}(t) = (5t)\hat{i} + (12\cos t)\hat{j} + (12\sin t)\hat{k}$.

(4 points) *a.* Find the unit normal vector $\hat{N}(t)$.

$$r'(t) = 5\hat{i} + (-12\sin t)\hat{i} + (12\cos t)\hat{k}$$

$$|\vec{r}'(t)| = \sqrt{25 + 144\sin^2 t + 144\cos^2 t} = \sqrt{25+144} = \sqrt{169} = 13$$

$$\hat{T}(t) = \frac{5}{13}\hat{i} + \left(-\frac{12}{13}\sin t\right)\hat{j} + \left(\frac{12}{13}\cos t\right)\hat{k}$$

$$\hat{T}'(t) = \left(-\frac{12}{13}\cos t\right)\hat{j} + \left(-\frac{12}{13}\sin t\right)\hat{k}$$

$$|\hat{T}'(t)| = \sqrt{\frac{144}{189}\cos^2 t + \frac{144}{169}\sin^2 t} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

$$\hat{N}(t) = (-\cos t)\hat{j} + (-\sin t)\hat{k}$$

(4 points) *b.* Find the binormal vector $\hat{B}(t)$.

$$\hat{B} = \hat{T} \times \hat{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{5}{13} & -\frac{12}{13}\sin t & \frac{12}{13}\cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = \hat{i}\left(\frac{12}{13}\sin^2 t + \frac{12}{13}\cos^2 t\right) - \hat{j}\left(-\frac{5}{13}\sin t\right) + \hat{k}\left(-\frac{5}{13}\cos t\right)$$

$$= \frac{12}{13}\hat{i} + \frac{5}{13}\sin t\,\hat{j} - \frac{5}{13}\cos t\,\hat{k}$$

(4 points) *4.* Solve the following initial value problem for $\vec{r}(t)$:

$$\vec{v}(t) = (t^3 + 4t)\hat{i} + t\,\hat{j} + 2t^2\hat{k} \text{ with } \vec{r}(0) = \hat{i} + 2\hat{j}$$

$\vec{r}(t) = \left(\frac{t^4}{4} + 2t^2\,\hat{i} + \frac{t^2}{2}\hat{j} + \frac{2}{3}t^3\hat{k}\right) + \vec{C}$

$\vec{r}(0) = \vec{C} = \hat{i} + 2\hat{j}$

$\vec{r}(t) = \left(\frac{t^4}{4} + 2t^2 + 1\right)\hat{i} + \left(\frac{t^2}{2} + 2\right)\hat{j} + \left(\frac{2}{3}t^3\right)\hat{k}$

5. Given the following function: $f(x,y) = \sqrt{25 - x^2 - y^2}$

(2 points) *a.* Find the function's domain.

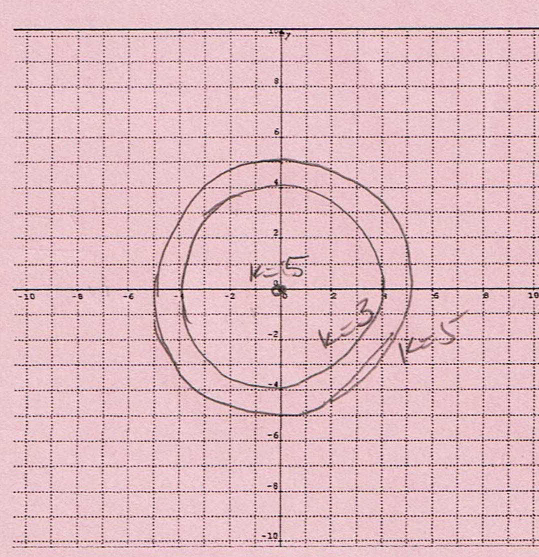
$\{(x,y) \mid 25 - x^2 - y^2 \geq 0\}$

or $\{(x,y) \mid x^2 + y^2 \leq 25\}$

(2 points) *b.* Find the function's range.

$0 \leq z \leq 5$

(3 points) *c.* Sketch the function's level curves when $k = 0$, $k = 3$, and $k = 5$.



$k=0$ $\sqrt{25 - x^2 - y^2} = 0$

$25 - x^2 - y^2 = 0$

$x^2 + y^2 = 25$

$k=3$ $\sqrt{25 - x^2 - y^2} = 3$

$25 - x^2 - y^2 = 9$

$16 = x^2 + y^2$

$k=5$

$\sqrt{25 - x^2 - y^2} = 5$

$25 - x^2 - y^2 = 25$

$x^2 + y^2 = 0$

(4 points) 6. Show that $f(x,y)=\dfrac{x^6}{x^6-y^2}$ has no limit as $(x,y)\to(0,0)$. Use $y=kx^3$, $(k\neq 1)$, and <u>explain</u> the result.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^3,\, k\neq 1}} \frac{x^6}{x^6-y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=kx^3,\, k\neq 1}} \frac{x^6}{x^6-(kx^3)^2} = \frac{1}{1-k^2},\ k\neq 1$$

Since the limit depends on k only, the limit does not exist (i.e., it is dependent on path)

or if $k=2$, the limit $= -1/3$
and if $k=3$, the limit $= -1/8$
so the limits are equal given two different paths. ∴ The limit does not exist.

7. Find the following limits.

(4 points) a. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4}$

$$= \lim_{(x,y)\to(2,2)} \frac{x-y}{(x^2+y^2)(x^2-y^2)} = \lim_{(x,y)\to(2,2)} \frac{x-y}{(x^2+y^2)(x+y)(x-y)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{1}{(x^2+y^2)(x+y)} = \frac{1}{(2^2+2^2)(2+2)}$$

$$= \frac{1}{(8)(4)} = \frac{1}{32}$$

(4 points) b. $\displaystyle\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} \cdot \frac{\sqrt{x}+\sqrt{y+1}}{\sqrt{x}+\sqrt{y+1}}$

$$= \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{x-(y+1)}{(x-y-1)(\sqrt{x}+\sqrt{y+1})} = \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{x-y-1}{(x-y-1)(\sqrt{x}+\sqrt{y+1})}$$

$$= \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{1}{\sqrt{x}+\sqrt{y+1}} = \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2} = \frac{1}{4}$$

8. Given $f(x,y) = x^2 e^{x^2y^2}$. Determine the following.

(2 points) a. $f_x = 2xe^{x^2y^2} + x^2 e^{x^2y^2} 2xy^2$

$= 2xe^{x^2y^2} + 2x^3y^2e^{x^2y^2}$

(2 points) b. $f_y = 2x^4 y e^{x^2y^2}$

(3 points) c. $f_{xx} = 2e^{x^2y^2} + 2xe^{x^2y^2} 2xy^2 + 6x^2y^2e^{x^2y^2} + 2x^3y^2e^{x^2y^2} 2xy^2$

$= 2e^{x^2y^2} + 4x^2y^2e^{x^2y^2} + 6x^2y^2e^{x^2y^2} + 4x^4y^4e^{x^2y^2}$

$= 2e^{x^2y^2} + 10x^2y^2e^{x^2y^2} + 4x^4y^4e^{x^2y^2}$

(3 points) d. $f_{yx} = 8x^3ye^{x^2y^2} + 2x^4ye^{x^2y^2} 2xy^2$

$= 8x^3ye^{x^2y^2} + 4x^5y^3e^{x^2y^2}$

(3 points) e. $f_{yy} = 2x^4e^{x^2y^2} + 2x^4ye^{x^2y^2} 2x^2y$

$= 2x^4e^{x^2y^2} + 4x^6y^2e^{x^2y^2}$

(6 points) *9.* Find the equations for the tangent plane and the normal line to the graph of the given equation at the point P_0.

$$x^2+y^2-z^2=18; \quad P_0(3,5,-4)$$

$\vec{\nabla} f = 2x\hat{i} + 2y\hat{j} - 2z\hat{j}$

$\vec{\nabla} f|_{(3,5,-4)} = 6\hat{i} + 10\hat{j} + 8\hat{j}$

$6(x-3)+10(y-5)+8(z+4)=0$

$6x-18+10y-50+8z+32=0$

$6x+10y+8z-36=0$

$x=3+6t$

$y=5+10t$

$z=-4+8t$

(6 points) *10.* Given $r=y^2+xzw^3$, $x=3t^2-2$, $y=t^2$, $z=4t^3+3$. $w=5t+2$. Write the chain rule that is used to find $\frac{dr}{dt}$ and then use the chain rule to find $\frac{dr}{dt}$.

$$\frac{dr}{dt} = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt} + \frac{\partial r}{\partial z}\frac{dz}{dt} + \frac{\partial r}{\partial w}\frac{dw}{dt}$$

$$\frac{dr}{dt} = (zw^3)(6t) + (2y)(2t) + (xw^3)(12t^2) + (3xzw^2)(5)$$

$$= (4t^3+3)(5t+2)^3(6t) + 2t^2(2t) + (3t^2-2)(5t+2)^3(12t^2) + 3(3t^2-2)(4t^3+3)(5t+2)^2(5)$$

$$= (6t)(4t^3+3)(5t+2)^3 + 4t^3 + (12t^2)(3t^2-2)(5t+2)^3 + 15(3t^2-2)(4t^3+3)(5t+2)^3$$

(2 points) *11.* Write the chain rules that are used to find $\partial w/\partial x$ and $\partial w/\partial y$ if $w=f(r,s)$, $r=g(x,y)$ $s=h(x,y)$.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y}$$

(6 points) *12.* Find the directional derivative of the following function at P_0 in the direction of $\bar{v}$:

$$f(x,y)=x^2e^{-2y};\quad P_0(1,0);\quad \bar{v}=\hat{i}+\hat{j}$$

$\vec{\nabla} f = 2xe^{-2y}\hat{\imath} - 2x^2e^{-2y}\hat{\jmath}$ $\quad |\vec{v}| = \sqrt{2}$

$\vec{\nabla} f|_{P_0} = 2\hat{\imath} - 2\hat{\jmath}$ $\quad \hat{v} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$

$\vec{\nabla} f|_{P} \cdot \hat{v} = (2\hat{\imath} - 2\hat{\jmath})\cdot(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$

$= \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} = 0$

13. Given the following: $f(x,y)=x^2y+\sqrt{y};\ P_0(2,1)$

(2 points) *a.* State the direction in which the function increases most rapidly at P_0.

$\vec{\nabla} f = 2xy\hat{\imath} + (x^2 + \frac{1}{2}y^{-1/2})\hat{\jmath}$ $\quad |\vec{\nabla} f| = \sqrt{16 + \frac{81}{4}} = \sqrt{\frac{64+81}{4}} = \frac{\sqrt{145}}{2}$

$\vec{\nabla} f|_P = 4\hat{\imath} + (4 + \frac{1}{2})\hat{\jmath}$

$= 4\hat{\imath} + (\frac{9}{2})\hat{\jmath}$

$\hat{u} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{4}{\sqrt{145}/2}\hat{\imath} + \frac{9/2}{\sqrt{145}/2}\hat{\jmath}$

$= \frac{8}{\sqrt{145}}\hat{\imath} + \frac{9}{\sqrt{145}}\hat{\jmath}$

(2 points) *b.* State the direction in which the function decreases most rapidly at P_0.

$-\hat{u} = -\frac{8}{\sqrt{145}}\hat{\imath} - \frac{9}{\sqrt{145}}\hat{\jmath}$

(2 points) *c.* State the directions in which the function has zero change at P_0.

$\hat{z}_1 = \frac{9}{\sqrt{145}}\hat{\imath} - \frac{8}{\sqrt{145}}\hat{\jmath}$ $\quad \hat{z}_2 = -\frac{9}{\sqrt{145}}\hat{\imath} + \frac{8}{\sqrt{145}}\hat{\jmath}$

(6 points) *14.* Use partial derivatives to find $\dfrac{dy}{dx}$ if $e^{x^2y^2}+4x^3=5x^3y+3\cos y$, where y is a differentiable function of x.

$F(x,y)=e^{x^2y^2}+4x^3-5x^3y-3\cos y=0$

$F_x=2xy^2e^{x^2y^2}+12x^2-15x^2y$

$F_y=2x^2ye^{x^2y^2}-5x^3+3\sin y$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{2xy^2e^{x^2y^2}+12x^2-15x^2y}{2x^2ye^{x^2y^2}-5x^3+3\sin y}$$

(6 points) *15.* Find the standard linearization $L(x,y)$ of the function at the given point.

$$f(x,y)=e^{2y-x} \text{ at } (1,2)$$

$f_x=-e^{2y-x}$ $\quad f_y=2e^{2y-x}$ $\quad f(1,2)=e^3$

$f_x(1,2)=-e^3$ $\quad f_y=2e^3$

$-e^3(x-1)+2e^3(y-2)=z-e^3$ $\quad e^3[1-x+1+2y-4]=z$

$e^3-e^3(x-1)+2e^3(y-2)=z$ $\quad e^3[-x+2y-2]=z$

$e^3[1-(x-1)+2(y-2)]=z$

(3 points) *16.* Given $A=2\pi r^2+2\pi rh$. Write the total differential dA.

$dA=(4\pi r+2\pi h)\,dr+2\pi r\,dh.$

(5 points) *17.* Given $f(x,y) = xy - 2x - 2y - x^2 - y^2$. Find the critical points of *f*.

$f_x = y - 2 - 2x \qquad f_y = x - 2 - 2y$

$y - 2 - 2x = 0$ AND $x - 2 - 2y = 0$

$-2x + y = 2$

$2(x - 2y = 2)$

$$\begin{array}{r} -2x + y = 2 \\ 2x - 4y = 4 \\ \hline -3y = 6 \\ y = -2 \end{array}$$

$x - 2(-2) = 2$

$x + 4 = 2$

$x = -2$

$(-2, -2)$

Optional Extra Credit Question:

(4 points) *18.* Use the limit definition to find the directional derivative of $f(x,y) = x^2 + 3y^2$ at the point $P_0(2,-1)$ in the direction of $\vec{u} = 3\hat{i} - 4\hat{j}$

$|\vec{u}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$

$\hat{u} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$

$$\lim_{h \to 0} \frac{f(2 + \frac{3}{5}h, -1 - \frac{4}{5}h) - f(2,-1)}{h} = \lim_{h \to 0} \frac{\left[(2 + \frac{3}{5}h)^2 + 3(-1 - \frac{4}{5}h)^2\right] - \left[(2)^2 + 3(-1)^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{4 + \frac{12}{5}h + \frac{9}{25}h^2 + 3\left(1 + \frac{8}{5}h + \frac{16}{25}h^2\right) - 7}{h}$$

$$= \lim_{h \to 0} \frac{4 + \frac{12}{5}h + \frac{9}{25}h^2 + 3 + \frac{24}{5}h + \frac{48}{25}h^2 - 7}{h}$$

$$= \lim_{h \to 0} \frac{\frac{36}{5}h + \frac{57}{25}h^2}{h} = \lim_{h \to 0} \left(\frac{36}{5} + \frac{57}{25}h\right) = \frac{36}{5}$$