MATH 280 – EXAM #2 **Name:** KEY
Winter Session 2020

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There are 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. Good luck!

1. Given that the following position vector describes a projectile in motion where distance is in feet and time is in seconds: $\vec{r}(t)=\left(128\sqrt{3}\right)t\hat{i}+\left(144+128t-16t^2\right)\hat{j}$

(3 points) *a.* What is the maximum height that the projectile attains?

$r_y = 144 + 128t - 16t^2$

$r_y' = 128 - 32t$

$128 - 32t = 0$

$\frac{128}{32} = \frac{32t}{32}$

$t = 4 \text{ sec}$

$r_y(4) = 144 + 128(4) - 16(4)^2$

$= 144 + 512 - 256$

$= 400 \text{ ft}$

(3 points) *b.* When does the projectile hit the ground?

$144 + 128t - 16t^2 = 0$

$-16(t^2 - 8t - 9) = 0$

$-16(t-9)(t+1) = 0$

$t = 9 \text{ sec}$

(3 points) *c.* What is the range of the object?

$r_x(9) = (128\sqrt{3})(9) = 1152\sqrt{3} \text{ ft}$

(4 points) 2. A baseball is thrown at a speed of 10 ft/s from the stands 15 ft above the field at an angle of $30°$ up from the horizontal. What is the position vector that models this situation? $\left(g = 32 \text{ ft/s}^2\right)$

$\vec{r}(t) = (10\cos 30°)t\,\hat{i} + \left(-\frac{1}{2}(32)t^2 + (10\sin 30°)t + 15\right)\hat{j}$

$= (5\sqrt{3})\,t\,\hat{i} + \left(-16t^2 + 5t + 15\right)\hat{j}$

3. Determine the domain of the following functions.

(2 points) a. $f(x,y) = \ln\left(x^2 + y^2 - 1\right)$ $\{(x,y) \mid x^2+y^2-1>0\}$

(2 points) b. $f(x,y) = \dfrac{\sqrt{x-y+1}}{y-1}$ $\{(x,y) \mid x-y+1 \geq 0,\ y \neq 1\}$

(3 points) 4. Given the following function: $f(x,y) = \sqrt{25 - x^2 - y^2}$. Sketch the function's level curves when $k = 0$, $k = 4$, and $k = 5$.

$K=0$

$\sqrt{25-x^2-y^2} = 0$

$25 - x^2 - y^2 = 0$

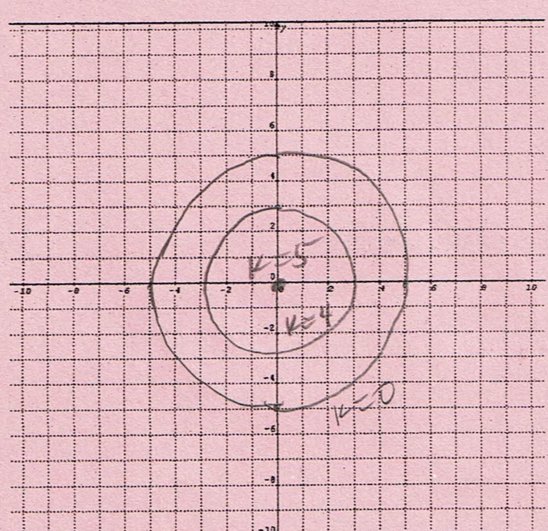
$x^2 + y^2 = 25$

$K=4$

$\sqrt{25-x^2-y^2} = 4$

$25 - x^2 - y^2 = 16$

$9 = x^2 + y^2$



$K=5$

$\sqrt{25-x^2-y^2} = 5$

$25 - x^2 - y^2 = 25$

$x^2 + y^2 = 0$

(4 points) *5.* Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2}$ does not exist by using the two paths approach.

x-axis $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^4-y^2}{x^4+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^4}{x^4} = 1$

y-axis $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^4-y^2}{x^4+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ x=0}} -\frac{y^2}{y^2} = -1$

Since the limits are different along two different paths, the limit does not exist.

6. Find the following limits.

(4 points) *a.* $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x-y}{x^2-y^2}$ $\displaystyle = \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x-y}{(x+y)(x-y)} = \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{1}{x+y} = \frac{1}{1+1} = \frac{1}{2}$

(4 points) *b.* $\displaystyle\lim_{\substack{(x,y)\to(9,7)\\ x+y\neq 16}} \frac{x+y-16}{\sqrt{x+y}-4}$ $\displaystyle \cdot \frac{\sqrt{x+y}+4}{\sqrt{x+y}+4} = \lim_{\substack{(x,y)\to(9,7)\\ x+y\neq 16}} \frac{(x+y-16)(\sqrt{x+y}+4)}{x+y-16}$

$\displaystyle = \lim_{\substack{(x,y)\to(9,7)\\ x+y\neq 16}} \sqrt{x+y}+4 = \sqrt{9+7}+4 = \sqrt{16}+4 = 4+4 = 8$

7. Given $f(x,y) = e^{xy}\sin y$. Determine the following.

(3 points) *a.* $f_x = e^{xy} y \sin y$

(3 points) *b.* $f_y = xe^{xy}\sin y + e^{xy}\cos y$

(3 points) *c.* $f_{xx} = e^{xy} y^2 \sin y$

(3 points) *d.* $f_{yx} = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y$

(3 points) *e.* $f_{yy} = x^2e^{xy}\sin y + xe^{xy}\cos y + xe^{xy}\cos y - e^{xy}\sin y$

$= x^2 e^{xy}\sin y + 2xe^{xy}\cos y - e^{xy}\sin y$

(6 points) *8.* Find the equations for the tangent plane and the normal line at the point P_0 on the following surface.

$$x^2 + 2y^2 - 3z^2 = 3; \quad P_0(2,-1,1)$$

$\vec{\nabla} f = 2x\hat{\imath} + 4y\hat{\jmath} - 6z\hat{k}$

$\vec{\nabla} f|_{P_0} = 4\hat{\imath} - 4\hat{\jmath} - 6\hat{k}$

$4(x-2) - 4(y+1) - 6(z-1) = 0$

$4x - 8 - 4y - 4 - 6z + 6 = 0$

$4x - 4y - 6z - 6 = 0$

$2x - 2y - 3z = 3$

$x = 2 + 4t$

$y = -1 - 4t$

$z = 1 - 6t$

(4 points) *9.* Given $w = y^3 + x^2 z$, $x = 2t^3$, $y = t^2 + 3$, $z = 3t + 2$. Write the chain rule that is used to find $\frac{dw}{dt}$ and then use the chain rule to find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$= (2xz)(6t^2) + (3y^2)(2t) + (x^2)(3)$

$= (2)(2t^3)(3t+2)(6t^2) + 3(t^2+3)^2(2t) + 3(2t^3)^2$

$= 24t^5(3t+2) + 6t(t^2+3)^2 + 12t^6$

$= 72t^6 + 48t^5 + 6t(t^4 + 6t^2 + 9) + 12t^6$

$= 72t^6 + 48t^5 + 6t^5 + 36t^3 + 54t + 12t^6 = 84t^6 + 54t^5 + 36t^3 + 54t$

(2 points) *10.* Write the chain rules that are used to find $\partial w/\partial x$ and $\partial w/\partial y$ if $w = f(r,s,t,u)$, $r = g(x,y)$ $s = h(x,y)$, $t = k(x,y)$, $u = l(x,y)$.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial y}$$

(6 points) *11.* Find the directional derivative of the following function at P_0 in the direction of $\bar{u}$:

$$f(x,y,z) = x^2 + 2y^2 - 3z^2; \quad P_0(1,1,1); \quad \bar{u} = \hat{i} + \hat{j} + \hat{k}$$

$\vec{\nabla} f = 2x\hat{i} + 4y\hat{j} - 6z\hat{k}$ $\quad |\vec{u}| = \sqrt{3}$ $\quad \hat{u} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

$\vec{\nabla} f|_p = 2\hat{i} + 4\hat{j} - 6\hat{k}$

$\vec{\nabla} f \cdot \hat{u} = (2\hat{i} + 4\hat{j} - 6\hat{k}) \cdot \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$

$= \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{6}{\sqrt{3}} = 0$

12. Given the following: $f(x,y) = x^2 y + e^{xy}\sin y; \; P_0(1,0)$

(3 points) *a.* State the direction in which the function increases most rapidly at P_0.

$\vec{\nabla} f = (2xy + ye^{xy}\sin y)\hat{i} + (x^2 + xe^{xy}\sin y + e^{xy}\cos y)\hat{j}$

$\vec{\nabla} f|_{P_0} = 0\hat{i} + 2\hat{j}$ $\qquad$ Direction: $\hat{j}$

(3 points) *b.* State the direction in which the function decreases most rapidly at P_0.

Decreases most rapidly: $-\hat{j}$

(3 points) *c.* State the directions in which the function has zero change at P_0.

zero change $\hat{i}, -\hat{i}$

(6 points) *13.* Use partial derivatives to perform implicit differentiation to find $\frac{dy}{dx}$ if $4x^3y^4+\cos(x^2y^3)=x^2\sin(xy)$, where y is a differentiable function of x.

$F(x,y)=4x^3y^4+\cos(x^2y^3)-x^2\sin(xy)=0$

$F_x=12x^2y^4-2xy^3\sin(x^2y^3)-2x\sin(xy)-x^2y\cos(xy)$

$F_y=16x^3y^3-3x^2y^2\sin(x^2y^3)-x^3\cos(xy)$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{12x^2y^4-2xy^3\sin(x^2y^3)-2x\sin(xy)-x^2y\cos(xy)}{16x^3y^3-3x^2y^2\sin(x^2y^3)-x^3\cos(xy)}$$

(6 points) *14.* Find the standard linearization $L(x,y)$ of the function at the given point.

$$f(x,y)=1+x\ln(xy-5) \text{ at } (2,3)$$

$f_x=\ln(xy-5)+\frac{xy}{xy-5}$ $\quad f_y=\frac{x^2}{xy-5}$

$f_x|_{P_0}=\frac{6}{6-5}=6$ $\quad f_y|_{P_0}=\frac{4}{6-5}=4$ $\quad f(2,3)=1+2(0)=1$

$L(x,y)=1+6(x-2)+4(y-3)$

$=1+6x-12+4y-12$

$=6x+4y-23$

(8 points) *15.* Use Lagrange multipliers to determine the maximum and minimum values of $f(x,y,z)=x-2y+5z$ subject to the constraint $x^2+y^2+z^2=30$.

$\vec{\nabla}f=\hat{i}-2\hat{j}+5\hat{k}$ $\quad \vec{\nabla}g=2x\hat{i}+2y\hat{j}+2z\hat{k}$

$1=2x\lambda$ $\quad -2=2y\lambda$ $\quad 5=2z\lambda$ $\quad x^2+y^2+z^2=30$

$\frac{1}{2\lambda}=x$ $\quad \frac{-2}{2\lambda}=y$ $\quad \frac{5}{2\lambda}=z$ $\quad \frac{1}{4\lambda^2}+\frac{4}{4\lambda^2}+\frac{25}{4\lambda^2}=30$

$\frac{30}{4\lambda^2}=30$

$\lambda=\frac{1}{2}$ $\quad x=1$ $\quad y=-2$ $\quad z=5$

$\lambda=-\frac{1}{2}$ $\quad x=-1$ $\quad y=2$ $\quad z=-5$

$1=4\lambda^2$

$\frac{1}{4}=\lambda^2$

$\lambda=\pm\frac{1}{2}$

$f(1,-2,5)=1+4+25=30$ maximum

$f(-1,2,-5)=-1-4-25=-30$ minimum.

(8 points) *16.* Given $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$. Find all local maxima, local minima, and saddle points.

$f_x = 12x - 6x^2 + 6y$ $\quad$ $f_y = 6y + 6x$

$12x - 6x^2 + 6y = 0$ $\quad$ $6y + 6x = 0$

$y = -x$

$12x - 6x^2 - 6x = 0$

$6x - 6x^2 = 0$

$6x(1-x) = 0$

$x = 0, 1$ $\quad$ $x=0,\ y=0$

$x = 1 \quad y = -1$

$(0,0)$ $(1,-1)$ critical points.

$f_{xx} = 12 - 12x$ $\quad$ $f_{yy} = 6$ $\quad$ $f_{xy} = 6$

$D(0,0) = (12)(6) - (6)^2 = 72 - 36 = 36 > 0$ $\quad$ $f_{xx} = 12 > 0$ local minimum

$D(1,-1) = (0)(6) - (6)^2 = -36 < 0$ $\quad$ saddle point

$(0,0)$ local minimum

$(1,-1)$ saddle point.