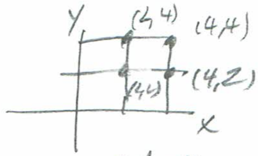
Exam #3

3pts

1.) $\iint_R (x^2y^2+2y)\,dA \quad R=[0,4]\times[0,4]$



$\Delta A = 4$

$V \approx f(2,2)\Delta A + f(2,4)\Delta A + f(4,2)\Delta A + f(4,4)\Delta A$

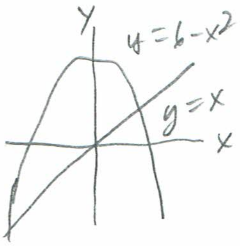
$V \approx 20\cdot 4 + 72\cdot 4 + 68\cdot 4 + 264\cdot 4$

$V \approx 80 + 288 + 272 + 1056$

$V \approx 1696$

5pts.

2.)



$6-x^2=x$

$0=x^2+x-6$

$0=(x+3)(x-2)$

$x=-3,2$

$\int_{-3}^{2}\int_{x}^{6-x^2} x^2\,dy\,dx = \int_{-3}^{2} x^2y\Big|_{x}^{6-x^2}dx = \int_{-3}^{2} x^2[6-x^2-x]\,dx$

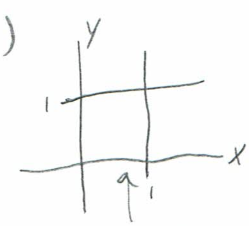
$=\int_{-3}^{2}(6x^2-x^4-x^3)\,dx = \left(2x^3-\frac{x^5}{5}-\frac{x^4}{4}\right)\Big|_{-3}^{2}$

$=\left(16-\frac{32}{5}-4\right)-\left(-54+\frac{243}{5}-\frac{81}{4}\right) = 66+\left(\frac{-32-243}{5}\right)+\frac{81}{4}$

$=66-\frac{275}{5}+\frac{81}{4} = 66-55+\frac{81}{4} = 11+\frac{81}{4} = \frac{44+81}{4} = \frac{125}{4}$

5pts.

3.)



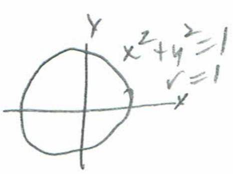
Area = 1

$\int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 x\frac{y^2}{2}\Big|_0^1 dx = \int_0^1 \frac{1}{2}x\,dx = \frac{x^2}{4}\Big|_0^1 = \frac{1}{4}$

Average value $= \dfrac{\iint_R f(x,y)\,dA}{\text{Area of } R} = \dfrac{1/4}{1} = 1/4$

6pts.

4.)



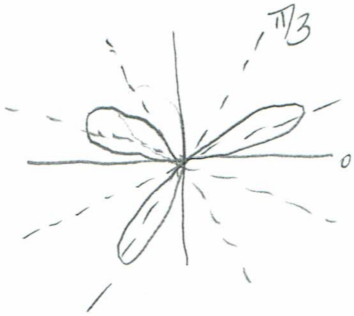
$\iint_R 9-x^2-y^2\,dA = \int_0^{2\pi}\int_0^1 (9-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (9r-r^3)\,dr\,d\theta$

$=\int_0^{2\pi}\left(\frac{9r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 d\theta = \int_0^{2\pi}\left(\frac{9}{2}-\frac{1}{4}\right)d\theta = \frac{17}{4}\int_0^{2\pi}d\theta = \frac{17}{4}\theta\Big|_0^{2\pi} = \frac{17}{4}(2\pi) = \frac{17\pi}{2}$

6pts

5.) $r = 4\sin 3\theta$ period $= \frac{2\pi}{3}$

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
r	0	4	0	-4	0	4	0	-4	0	4	0	-4	0



$$\int_0^{\pi/3}\int_0^{4\sin 3\theta} r\,dr\,d\theta = \int_0^{\pi/3} \frac{r^2}{2}\Big|_0^{4\sin 3\theta} d\theta = \frac{1}{2}\int_0^{\pi/3} (4\sin 3\theta)^2 d\theta$$

$$= 8\int_0^{\pi/3} \sin^2 3\theta\, d\theta = 8\int_0^{\pi/3}\left(\frac{1-\cos 6\theta}{2}\right) d\theta = 4\left(\theta - \frac{\sin 6\theta}{6}\right)\Big|_0^{\pi/3}$$

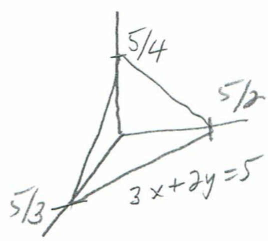
$$= 4\left[(\pi/3 - 0) - (0-0)\right] = 4\pi/3$$

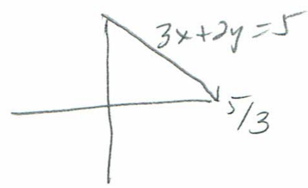
5pts.

6.) $3x + 2y + 4z = 5$

$4z = 5 - 3x - 2y$

$z = \frac{5}{4} - \frac{3}{4}x - \frac{1}{2}y$





$2y = -3x + 5$

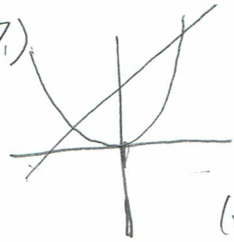
$y = -\frac{3}{2}x + \frac{5}{2}$

$$\int_0^{5/3}\int_0^{-\frac{3}{2}x + 5/2} \sqrt{1 + \left(-\frac{3}{4}\right)^2 + \left(-\frac{1}{2}\right)^2}\, dy\,dx = \int_0^{5/3}\int_0^{-\frac{3}{2}x+\frac{5}{2}} \sqrt{1 + \frac{9}{16} + \frac{1}{4}}\, dy\,dx$$

$$= \frac{\sqrt{29}}{4}\int_0^{5/3}\int_0^{-3/2x+5/2} dy\,dx = \frac{\sqrt{29}}{4}\int_0^{5/3} y\Big|_0^{-\frac{3}{2}x+\frac{5}{2}} dx = \frac{\sqrt{29}}{4}\int_0^{5/3}\left(-\frac{3}{2}x + \frac{5}{2}\right)dx = \frac{\sqrt{29}}{4}\left(-\frac{3}{4}x^2 + \frac{5}{2}x\right)\Big|_0^{5/3}$$

$$= \frac{\sqrt{29}}{4}\left(-\frac{3}{4}\left(\frac{25}{9}\right) + \frac{5}{2}\left(\frac{5}{3}\right)\right) = \frac{\sqrt{29}}{4}\left(-\frac{75}{36} + \frac{25}{6}\right) = \frac{\sqrt{29}}{4}\left(\frac{75}{36}\right) = \frac{75\sqrt{29}}{144} = \frac{25\sqrt{29}}{48}$$

7.)



$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2,\ x = -1$

4pts a.)

$$M_x = \int_{-1}^{2}\int_{x^2}^{x+2} y\,kx^2\,dy\,dx = k\int_{-1}^{2} \frac{y^2}{2}x^2 \Big|_{x^2}^{x+2} dx = \frac{k}{2}\int_{-1}^{2} x^2\left[(x+2)^2 - (x^2)^2\right] dx$$

$$= \frac{k}{2}\int_{-1}^{2} x^2\left(x^2+4x+4-x^4\right) dx = \frac{k}{2}\int_{-1}^{2}\left(x^4+4x^3+4x^2-x^6\right) dx = \frac{k}{2}\left(\frac{x^5}{5} + x^4 + \frac{4}{3}x^3 - \frac{x^7}{7}\right)\Big|_{-1}^{2}$$

$$= \frac{k}{2}\left(\left(\frac{32}{5}+16+\frac{32}{3}-\frac{128}{7}\right) - \left(-\frac{1}{5}+1-\frac{4}{3}+\frac{1}{7}\right)\right) = \frac{k}{2}\left(\frac{33}{5}+15+\frac{36}{3}-\frac{129}{7}\right)$$

$$= \frac{k}{2}\left(\frac{531}{35}\right) = \frac{531}{70}k$$

4pts b.)

$$M_y = \int_{-1}^{2}\int_{x^2}^{x+2} x\,kx^2\,dy\,dx = k\int_{-1}^{2}\int_{x^2}^{x+2} x^3\,dy\,dx = k\int_{-1}^{2} x^3 y\Big|_{x^2}^{x+2} dx = k\int_{-1}^{2} x^3(x+2-x^2)\,dx = k\int_{-1}^{2}(x^4+2x^3-x^5)\,dx$$

$$= k\left(\frac{x^5}{5}+\frac{x^4}{2}-\frac{x^6}{6}\right)\Big|_{-1}^{2} = k\left(\left(\frac{32}{5}+8-\frac{32}{3}\right)-\left(-\frac{1}{5}+\frac{1}{2}-\frac{1}{6}\right)\right) = k\left(\frac{33}{5}+8-\frac{32}{3}-\frac{1}{2}+\frac{1}{6}\right) = \frac{18k}{5}$$

4pts c.)

$$M = \int_{-1}^{2}\int_{x^2}^{x+2} kx^2\,dy\,dx = k\int_{-1}^{2} x^2 y\Big|_{x^2}^{x+2} = k\int_{-1}^{2} x^2(x+2-x^2)\,dx = k\int_{-1}^{2}(x^3+2x^2-x^4)\,dx$$

$$= k\left(\frac{x^4}{4}+\frac{2}{3}x^3-\frac{x^5}{5}\right)\Big|_{-1}^{2} = k\left(\left(4+\frac{16}{3}-\frac{32}{5}\right)-\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{5}\right)\right) = k\left(4+6-\frac{33}{5}-\frac{1}{4}\right) = \frac{63k}{20}$$

2pts. d.)

$$\bar{x} = \frac{M_y}{M} = \frac{18k/5}{63k/20} = \frac{18}{5}\cdot\frac{20}{63} = \frac{72}{63} = \frac{8}{7}$$

$$\left(\frac{8}{7}, \frac{118}{49}\right)$$

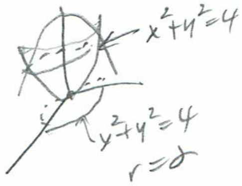
$$\bar{y} = \frac{M_x}{M} = \frac{\frac{531k}{70}}{\frac{63k}{20}} = \frac{\overset{59}{\cancel{531}}}{70}\cdot\frac{20}{\underset{7}{\cancel{63}}} = \frac{118}{49}$$

6pts.

8.) $8-x^2-y^2=x^2+y^2$

$8=2x^2+2y^2$

$4=x^2+y^2$



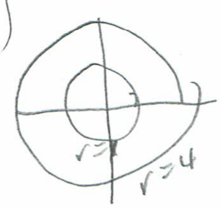
$$\iiint_E dz\,dy\,dx = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{8-r^2} dz\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} zr\Big|_{r^2}^{8-r^2} dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2} r(8-r^2-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}(8r-2r^3)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(4r^2-\frac{1}{2}r^4\right)\Big|_0^2 d\theta = \int_0^{2\pi}(16-8)\,d\theta = 8\theta\Big|_0^{2\pi} = 16\pi$$

7pts.

9.)



$$\int_0^{2\pi}\int_1^{4}\int_0^{r\sin\theta+4}(r\cos\theta - r\sin\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^{4} z(r^2\cos\theta - r^2\sin\theta)\Big|_0^{r\sin\theta+4} dr\,d\theta$$

$$=\int_0^{2\pi}\int_1^{4}(r\sin\theta+4)(r^2\cos\theta - r^2\sin\theta)\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_1^{4}(r^3\sin\theta\cos\theta - r^3\sin^2\theta + 4r^2\cos\theta - 4r^2\sin\theta)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(\frac{r^4}{4}\sin\theta\cos\theta - \frac{r^4}{4}\sin^2\theta + \frac{4}{3}r^3\cos\theta - \frac{4}{3}r^3\sin\theta\right)\Big|_1^4 d\theta$$

$$=\int_0^{2\pi}\left[\left(64\sin\theta\cos\theta - 64\sin^2\theta + \frac{256}{3}\cos\theta - \frac{256}{3}\sin\theta\right) - \left(\frac{1}{4}\sin\theta\cos\theta - \frac{1}{4}\sin^2\theta + \frac{4}{3}\cos\theta - \frac{4}{3}\sin\theta\right)\right]d\theta$$

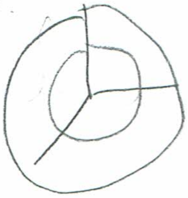
$$=\int_0^{2\pi}\left(\frac{255}{4}\sin\theta\cos\theta - \frac{255}{4}\sin^2\theta + \frac{764}{3}\cos\theta - \frac{764}{3}\sin\theta\right)d\theta$$

$$=\int_0^{2\pi}\left(\frac{255}{4}\sin\theta\cos\theta - \frac{255}{4}\left(\frac{1-\cos 2\theta}{2}\right) + \frac{764}{3}\cos\theta - \frac{764}{3}\sin\theta\right)d\theta$$

$$=\left(\frac{255}{4}\frac{\sin^2\theta}{2} - \frac{255}{8}\theta + \frac{255}{16}\sin 2\theta + \frac{764}{3}\sin\theta + \frac{764}{3}\cos\theta\right)\Big|_0^{2\pi}$$

$$=\left(-\frac{255}{8}(2\pi) + \frac{764}{3}\right) - \left(\frac{764}{3}\right) = -\frac{255\pi}{4}$$

7pts.
10



$$\int_0^{2\pi}\int_0^{\pi}\int_2^3 \left[(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2\right] \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_2^3 (\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^2 \sin^2\phi (\cos^2\theta + \sin^2\theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^2 \sin^2\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5} \sin^3\phi \Big|_2^3 d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{243}{5} - \frac{32}{5}\right) \sin^3\phi \, d\phi \, d\theta = \frac{211}{5}\int_0^{2\pi}\int_0^{\pi} \sin^3\phi \, d\phi \, d\theta$$

$$= \frac{211}{5}\int_0^{2\pi}\int_0^{\pi} \sin^2\phi \sin\phi \, d\phi \, d\theta = \frac{211}{5}\int_0^{2\pi}\int_0^{\pi} (1 - \cos^2\phi)\sin\phi \, d\phi \, d\theta$$

$$= \frac{211}{5}\int_0^{2\pi}\int_0^{\pi} (\sin\phi - \cos^2\phi \sin\phi) \, d\phi \, d\theta$$

$$= \frac{211}{5}\int_0^{2\pi} \left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Big|_0^{\pi} d\theta = \frac{211}{5}\int_0^{2\pi} \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] d\theta$$

$$= \frac{211}{5}\int_0^{2\pi} \frac{4}{3} \, d\theta = \frac{844}{15}\int_0^{2\pi} d\theta = \frac{844}{15}\theta \Big|_0^{2\pi} = \frac{1688\pi}{15}$$

11. $\displaystyle\int_0^{2/3}\int_0^{2-2y} (x+2y)\, e^{y-x}\, dx\,dy$

3pts

a $\quad u = x+2y \qquad v = x-y$

$x = u-2y \qquad v = u-2y-y \qquad x = u - 2\left(\frac{1}{3}u - \frac{1}{3}v\right)$

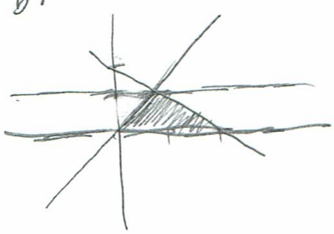
$v = u-3y \qquad x = u - \frac{2}{3}u + \frac{2}{3}v$

$3y = u-v$

$y = \frac{1}{3}u - \frac{1}{3}v \qquad x = \frac{1}{3}u + \frac{2}{3}v$

3pts.

b.



$x = 2-2y$

$2y = -x+2$

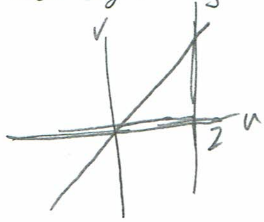
$y = -\frac{1}{2}x+1$

Boundaries: $y=0$, $y=x$, $x=2-2y$

$y=0$: $\quad \frac{1}{3}u - \frac{1}{3}v = 0 \Rightarrow \frac{1}{3}u = \frac{1}{3}v \Rightarrow u=v$

$y=x$: $\quad \cancel{\frac{1}{3}u} - \frac{1}{3}v = \cancel{\frac{1}{3}u} + \frac{2}{3}v \Rightarrow v=0$

$x=2-2y$: $\quad \frac{1}{3}u + \frac{2}{3}v = 2 - 2\left(\frac{1}{3}u - \frac{1}{3}v\right) \Rightarrow \frac{1}{3}u + \cancel{\frac{2}{3}v} = 2 - \frac{2}{3}u + \cancel{\frac{2}{3}v} \Rightarrow u=2$



2pts

c.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

3pts

d. $\displaystyle\int_0^{2}\int_0^{u} \frac{1}{3}\, u e^{-v}\, dv\,du$