MATH 280 Exam #3 KEY

7pts.

#1) $P = xyz$ $\quad x+y+z=100$

$\vec{\nabla}P = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$ $\quad \vec{\nabla}g = \hat{\imath}+\hat{\jmath}+\hat{k}$

$yz = \lambda \quad xz = \lambda \quad xy = \lambda \quad x+y+z=100$

$yz = xz$

$yz - xz = 0$

$z(y-x) = 0$

$z=0$ or $y=x$

Not positive

$xz = xy$

$xz - xy = 0$

$x(z-y) = 0$

$x=0$ ~~or~~ $z=y$

Not positive

$y+y+y=100$

$3y = 100$

$y = 100/3$

$x = 100/3$

$z = 100/3$

7pts

#2) $f(x,y,z) = 2x+2y+z \quad g(x,y,z) = x^2+y^2+z^2=9$

$\vec{\nabla}f = 2\hat{\imath}+2\hat{\jmath}+\hat{k} \quad \vec{\nabla}g = 2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k}$

$2 = 2x\lambda \quad 2 = 2y\lambda \quad 1 = 2z\lambda \quad x^2+y^2+z^2=9$

$x = \frac{1}{\lambda} \quad y = \frac{1}{\lambda} \quad z = \frac{1}{2\lambda}$

$\left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 9$

$\frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 9$

$\frac{4+4+1}{4\lambda^2} = 9$

$\frac{9}{4\lambda^2} = 9$

$1 = 4\lambda^2$

$\lambda^2 = \frac{1}{4}$

$\lambda = \pm\frac{1}{2}$

$\lambda = \frac{1}{2} \quad x=2 \quad y=2 \quad z=1$

$\lambda = -\frac{1}{2} \quad x=-2 \quad y=-2 \quad z=-1$

$f(2,2,1) = 2(2)+2(2)+1 = 9$

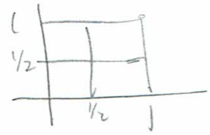
$f(-2,-2,-1) = 2(-2)+2(-2)+(-1) = -9$

max value $= 9$

min value $= -9$

5pts

#3) $f(x,y) = 2x^2y + 3y^2$



$\Delta A = \frac{1}{4}$

Area $\approx f(\frac{1}{2},\frac{1}{2})\,\Delta A + f(\frac{1}{2},1)\,\Delta A + f(1,\frac{1}{2})\,\Delta A + f(1,1)\,\Delta A$

Area $\approx \left(2(\frac{1}{2})^2\cdot\frac{1}{2} + 3(\frac{1}{2})^2\right)\cdot\frac{1}{4} + \left(2(\frac{1}{2})^2\cdot 1 + 3(1)^2\right)\cdot\frac{1}{4} + \left(2(1)^2\cdot(\frac{1}{2}) + 3(\frac{1}{2})^2\right)\cdot\frac{1}{4} + \left(2(1)^2\cdot 1 + 3(1)^2\right)\cdot\frac{1}{4}$

Area $\approx \left(\frac{1}{4} + \frac{3}{4}\right)\cdot\frac{1}{4} + \left(\frac{1}{2}+3\right)\cdot\frac{1}{4} + \left(1+\frac{3}{4}\right)\cdot\frac{1}{4} + (2+3)\cdot\frac{1}{4}$

Area $\approx \frac{1}{4} + \frac{7}{8} + \frac{7}{16} + \frac{5}{4}$

Area $\approx \frac{4+14+7+20}{16} = \frac{45}{16}$

7pts

#4.) $z = x^2,\quad y = 2-x^2,\quad y = x$

$2 - x^2 = x$

$0 = x^2 + x - 2$

$0 = (x+2)(x-1)$

$x = -2, 1$

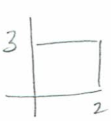
$$\int_{-2}^{1}\int_{x}^{2-x^2} x^2\,dy\,dx = \int_{-2}^{1}(x^2y)\Big|_{x}^{2-x^2}dx = \int_{-2}^{1} x^2\left[2-x^2-x\right]dx = \int_{-2}^{1}(2x^2 - x^4 - x^3)\,dx$$

$$= \left(\frac{2}{3}x^3 - \frac{x^5}{5} - \frac{x^4}{4}\right)\Big|_{-2}^{1} = \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4}\right)$$

$$= \frac{2}{3} - \frac{1}{5} - \frac{1}{4} + \frac{16}{3} - \frac{32}{5} + 4 = 10 - \frac{1}{4} - \frac{33}{5} = \frac{200-20-132}{20} = \frac{48}{20} = \frac{12}{5}$$

7pts.

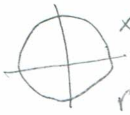
#5) $$\int_0^2\int_0^3 xy^2\,dy\,dx = \int_0^2 x\frac{y^3}{3}\Big|_0^3 dx = \int_0^2 9x\,dx = \frac{9}{2}x^2\Big|_0^2 = 18$$



Area $= 6$ Average Value $= \frac{18}{6} = 3$

7pts.

#6)



$x^2+y^2 \le 36$

$r^2 = 36$

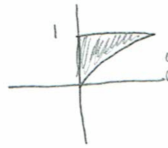
$r = 6$

$$\int_0^{2\pi}\int_0^{6} r^2\, r\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^6 d\theta = \int_0^{2\pi} \frac{6^4}{4}\,d\theta$$

$$= 324\int_0^{2\pi} d\theta = 324\,\theta\,\Big|_0^{2\pi} = 648\pi$$

7pts.

#7) $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\,dy\,dx$



$y = \sqrt{x}$

$x = y^2$

$$= \int_0^1 \int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 \sqrt{y^3+1}\,x\,\Big|_0^{y^2} dy = \int_0^1 y^2\sqrt{y^3+1}\,dy$$

$u = y^3+1$

$du = 3y^2\,dy$

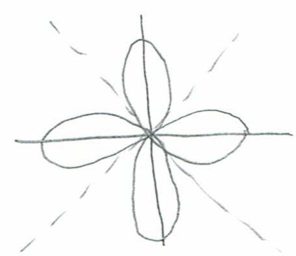
$\frac{1}{3}\,du = y^2\,dy$

$$\frac{1}{3}\int_0^1 u^{\frac{1}{2}}\,du = \frac{1}{3}\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{9}(y^3+1)^{3/2}\Big|_0^1 = \frac{2}{9}\left(2^{3/2}-1\right)$$

7pts

#8) $r = 4\cos 2\theta$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	4	0	-4	0	4	0	-4	0	4



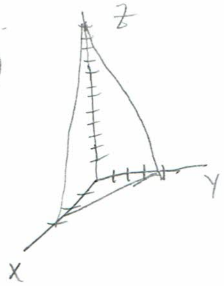
$$\int_{-\pi/2}^{\pi/2}\int_0^{4\cos 2\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^2}{2}\Big|_0^{4\cos 2\theta} d\theta =$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (4\cos 2\theta)^2\,d\theta = 8\int_{-\pi/2}^{\pi/2} \cos^2 2\theta\,d\theta = 8\int_{-\pi/2}^{\pi/2} \frac{1+\cos 4\theta}{2}\,d\theta = 4\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_{-\pi/2}^{\pi/2}$$

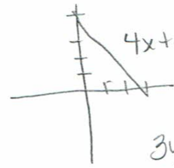
$$= 4\left(\frac{\pi}{2} - (-\pi/2)\right) = 4\pi$$

7pts.

#9.)



$4x+3y+z=12$



$4x+3y=12$

$3y=12-4x$

$y=4-\frac{4}{3}x$

$$\int_0^3\int_0^{4-\frac{4}{3}x}\int_0^{12-4x-3y} dz\,dy\,dx$$

$$=\int_0^3\int_0^{4-\frac{4}{3}x} z\Big|_0^{12-4x-3y} dy\,dx = \int_0^3\int_0^{4-\frac{4}{3}x}(12-4x-3y)\,dy\,dx = \int_0^3\left(12y-4xy-\frac{3}{2}y^2\right)\Big|_0^{4-\frac{4}{3}x} dx$$

$$=\int_0^3\left[12\left(4-\frac{4}{3}x\right)-4x\left(4-\frac{4}{3}x\right)-\frac{3}{2}\left(4-\frac{4}{3}x\right)^2\right]dx$$

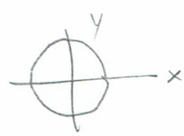
$$=\int_0^3\left[48-16x-16x+\frac{16}{3}x^2-\frac{3}{2}\left(16-\frac{32}{3}x+\frac{16}{9}x^2\right)\right]dx$$

$$=\int_0^3\left(48-32x+\frac{16}{3}x^2-24+16x-\frac{8}{3}x^2\right)dx = \int_0^3\left(24-16x+\frac{8}{3}x^2\right)dx$$

$$=24x-8x^2+\frac{8}{9}x^3\Big|_0^3 = 72-72+24=24$$

7pts.

#10) $\iiint_E z\,dV$ $\quad x^2+y^2=4$



$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{z^2}{2}r\Big|_{r^2}^4 dr\,d\theta = \int_0^{2\pi}\int_0^2\frac{r}{2}(16-r^4)\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^2(16r-r^5)\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(8r^2-\frac{r^6}{6}\right)\Big|_0^2 d\theta = \frac{1}{2}\int_0^{2\pi}\left(32-\frac{64}{6}\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left(32-\frac{32}{3}\right)d\theta$$

$$=\frac{1}{2}\left(\frac{96-32}{3}\right)\int_0^{2\pi}d\theta = \frac{32}{3}\theta\Big|_0^{2\pi} = \frac{64\pi}{3}$$

7pts.

#11) $\iiint_E y^2 z^2 \, dV$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\rho\sin\phi\sin\theta)^2(\rho\cos\phi)^2\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin^2\phi\sin^2\theta\,\rho^2\cos^2\phi\,\rho^2\sin\phi \quad d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^6\sin^3\phi\cos^2\phi\sin^2\theta \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^7}{7}\sin^3\phi\cos^2\phi\sin^2\theta \Big|_0^1 \, d\phi\, d\theta$$

$$= \frac{1}{7}\int_0^{2\pi}\int_0^{\pi/4} \sin^3\phi\cos^2\phi\sin^2\theta \, d\phi\, d\theta = \frac{1}{7}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\sin^2\phi\cos^2\phi\sin^2\theta \, d\phi\, d\theta$$

$$= \frac{1}{7}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi(1-\cos^2\phi)\cos^2\phi \sin^2\theta \, d\phi\, d\theta$$

$u = \cos\phi$
$du = -\sin\phi\, d\phi$
$-du = \sin\phi\, d\phi$

$$= -\frac{1}{7}\int_0^{2\pi}\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^{\pi/4}\sin^2\theta\, d\theta$$

$$= -\frac{1}{7}\int_0^{2\pi}\left(\frac{\cos^3\phi}{3} - \frac{\cos^5\phi}{5}\right)\Big|_0^{\pi/4}\sin^2\theta\, d\theta$$

$$= -\frac{1}{7}\int_0^{2\pi}\left(\left[\frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^5\right] - \left[\left(\frac{1}{3} - \frac{1}{5}\right)\right]\right)\sin^2\theta\, d\theta$$

$$= -\frac{1}{7}\int_0^{2\pi}\left(\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{40} - \frac{1}{3} + \frac{1}{5}\right)\left(\frac{1-\cos 2\theta}{2}\right) d\theta = -\frac{1}{7}\left(\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{40} - \frac{1}{3} + \frac{1}{5}\right)\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$= -\frac{1}{7}\left(\frac{10\sqrt{2} - 3\sqrt{2} - 40 + 24}{120}\right)\pi = -\frac{\pi}{7}\left(\frac{7\sqrt{2} - 16}{120}\right) = \frac{\pi}{7}\left(\frac{16 - 7\sqrt{2}}{120}\right)$$