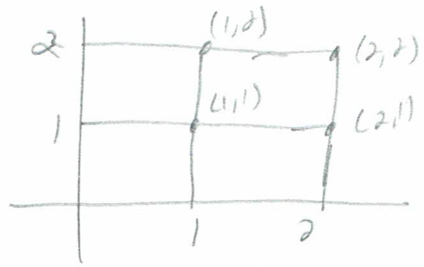
MATH 280 – EXAM #3 **Name:** KEY
Spring Semester 2018

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 103 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes or the textbook. You may not use any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam. You may not use a graphing calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(6 points) 1. Estimate $\iint\limits_R \left(1+xy^2\right)dA$ where $R=[0,2]\times[0,2]$ by dividing R into four equal squares and evaluating the function at the upper right corner of each square.

$\Delta A = 1$

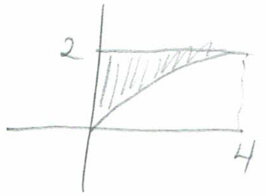


$f(1,1)\,\Delta A + f(2,1)\,\Delta A + f(1,2)\,\Delta A + f(2,2)\,\Delta A$

$= 2+3+5+9 = 19$

(4 points) 2. Sketch the region of integration and reverse the order of integration.

$$\int_0^4\int_{\sqrt{x}}^2 x\cos y^5\,dy\,dx$$



$$\int_0^2\int_0^{y^2} x\cos y^5\,dx\,dy$$

(10 points) 3. Evaluate the double integral $\iint_D (1-x+2y)\,dA$ where D is the region bounded by $x+y=1$ and $x^2+y=1$.

$y=1-x \quad y=1-x^2$

$1-x=1-x^2$

$x^2-x=0$

$x(x-1)=0$

$x=0,1$

$$\int_0^1\int_{1-x}^{1-x^2}(1-x+2y)\,dy\,dx=\int_0^1 (y-xy+y^2)\Big|_{1-x}^{1-x^2}dx$$

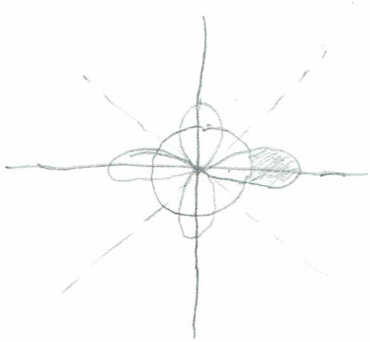
$$=\int_0^1\left[\left[1-x^2-x(1-x^2)+(1-x^2)^2\right]-\left[1-x-x(1-x)+(1-x)^2\right]\right]dx$$

$$=\int_0^1\left[(1-x^2-x+x^3+1-2x^2+x^4)-(1-x-x+x^2+1-2x+x^2)\right]dx$$

$$=\int_0^1\left[(2-x-3x^2+x^3+x^4)-(2-4x+2x^2)\right]dx=\int_0^1(3x-5x^2+x^3+x^4)\,dx$$

$$=\left(\frac{3x^2}{2}-\frac{5x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}\right)\Big|_0^1=\frac{90-100+15+12}{60}=\frac{17}{60}$$

(10 points) 4. Use a double integral to find the area of the region that lies outside the circle $r=2$ and inside one petal of $r=4\cos 2\theta$.



$4\cos 2\theta=2$

$\cos 2\theta=1/2$

$2\theta=\pi/3$

$\theta=\pi/6$

$$\int_{-\pi/6}^{\pi/6}\int_2^{4\cos 2\theta} r\,dr\,d\theta=\int_{-\pi/6}^{\pi/6}\frac{r^2}{2}\Big|_2^{4\cos 2\theta}d\theta$$

$$=\frac{1}{2}\int_{-\pi/6}^{\pi/6}(16\cos^2 2\theta-4)\,d\theta=2\int_{-\pi/6}^{\pi/6}\left[4\left(\frac{1+\cos 4\theta}{2}\right)-1\right]d\theta$$

$$=2\int_{-\pi/6}^{\pi/6}\left[2(1+\cos 4\theta)-1\right]d\theta=2\int_{-\pi/6}^{\pi/6}(1+\cos 4\theta)\,d\theta$$

$$=2\left(\theta+\frac{\sin 4\theta}{4}\right)\Big|_{-\pi/6}^{\pi/6}=2\left[\left(\frac{\pi}{6}+\frac{\sin\frac{2\pi}{3}}{4}\right)-\left(-\frac{\pi}{6}+\frac{\sin(-\frac{2\pi}{3})}{4}\right)\right]$$

$$=2\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right]=\frac{2\pi}{3}+\frac{\sqrt{3}}{2}$$

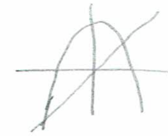
(10 points) 5. Use a triple integral to find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 2 - x^2$ and the line $y = x$, while the top of the solid is bounded by the paraboloid $z = x^2$.

$2 - x^2 = x$

$0 = x^2 + x - 2$

$0 = (x+2)(x-1)$

$x = -2, 1$



$$\int_{-2}^{1}\int_{x}^{2-x^2}\int_{0}^{x^2} dz\,dy\,dx$$

$$= \int_{-2}^{1}\int_{x}^{2-x^2} z\Big|_0^{x^2} dy\,dx = \int_{-2}^{1}\int_{x}^{2-x^2} x^2\,dy\,dx = \int_{-2}^{1} x^2 y\Big|_x^{2-x^2} dx = \int_{-2}^{1} x^2[2-x^2-x]\,dx$$

$$= \int_{-2}^{1}(2x^2-x^4-x^3)\,dx = \left(\frac{2x^3}{3}-\frac{x^5}{5}-\frac{x^4}{4}\right)\Big|_{-2}^{1} = \left(\frac{2}{3}-\frac{1}{5}-\frac{1}{4}\right)-\left(-\frac{16}{3}+\frac{32}{5}-\frac{16}{4}\right)$$

$$= \frac{2}{3}-\frac{1}{5}-\frac{1}{4}+\frac{16}{3}-\frac{32}{5}+\frac{16}{4} = 6+4-\frac{33}{5}-\frac{1}{4} = \frac{200-137}{20} = \frac{63}{20}$$

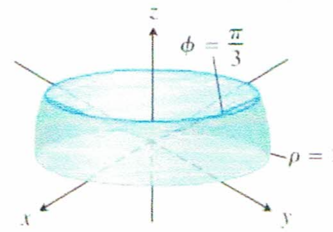
(10 points) 6. Use cylindrical coordinates to find the volume of the region bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by the xy-plane.

$$\int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} rz\Big|_0^{4-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r(4-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}(4r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_0^{2} d\theta = \int_0^{2\pi} 4\,d\theta = 4\theta\Big|_0^{2\pi} = 8\pi$$

(10 points) 7. Find the volume of the solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$ and above by the cone $\varphi = \pi/3$.



$$\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^2 \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \frac{\rho^3}{3}\sin\varphi \Big|_0^2 d\varphi\, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\sin\varphi\, d\varphi\, d\theta = \frac{8}{3}\int_0^{2\pi}(-\cos\varphi)\Big|_{\pi/3}^{\pi/2} d\theta$$

$$= \frac{8}{3}\int_0^{2\pi}\left[\left(-\cos\tfrac{\pi}{2}\right)-\left(-\cos\tfrac{\pi}{3}\right)\right] d\theta = \frac{8}{3}\cdot\frac{1}{2}\int_0^{2\pi} d\theta = \frac{4}{3}\theta\Big|_0^{2\pi} = \frac{8\pi}{3}$$

8. Given a thin plate of constant density $\delta = 3$ bounded by the x-axis and $y = \sin x$, $0 \le x \le \pi$.

(2 points) a. Set up the integral that would compute M_x.

$$\int_0^{\pi}\int_0^{\sin x} 3y\, dy\, dx$$

(2 points) b. Set up the integral that would compute M_y.

$$\int_0^{\pi}\int_0^{\sin x} 3x\, dy\, dx$$

(2 points) c. Set up the integral that would compute I_x.

$$\int_0^{\pi}\int_0^{\sin x} 3y^2\, dy\, dx$$

(2 points) c. Set up the integral that would compute I_y.

$$\int_0^{\pi}\int_0^{\sin x} 3x^2\, dy\, dx$$

(8 points) 9. Evaluate $\int_C 4x\,ds$ where C is the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.

$x = t \quad y = t^2 \quad 0 \le t \le 2$

$\vec{r}(t) = t\hat{i} + t^2\hat{j} \quad \vec{r}\,'(t) = \hat{i} + 2t\hat{j}$

$|\vec{r}\,'(t)| = \sqrt{1+4t^2}$

$\int_0^2 4t\sqrt{1+4t^2}\,dt$

$u = 1 + 4t^2$

$du = 8t\,dt$

$\frac{du}{8} = t\,dt$

$= \frac{1}{2}\int_0^2 u^{1/2}\,du$

$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^2 = \frac{1}{3}(1+4t^2)^{3/2}\Big|_0^2 = \frac{1}{3}(17^{3/2} - 1)$

(10 points) 10. Given $\vec{F} = (x+y)\hat{i} + (y-z)\hat{j} + z^2\hat{k}$ and the curve $\vec{r}(t) = t^2\hat{i} + t^3\hat{j} + t^2\hat{k}$, $0 \le t \le 1$. Find the work done by $\vec{F}$ over $\vec{r}(t)$.

$\vec{F} = (t^2+t^3)\hat{i} + (t^3 - t^2)\hat{j} + t^4\hat{k} \qquad d\vec{r} = 2t\hat{i} + 3t^2\hat{j} + 2t\hat{k}$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (2t^3 + 2t^4 + 3t^5 - 3t^4 + 2t^5)\,dt$

$= \int_0^1 (2t^3 - t^4 + 5t^5)\,dt$

$= \left(\frac{2t^4}{4} - \frac{t^5}{5} + \frac{5t^6}{6}\right)\Big|_0^1$

$= \frac{15 - 6 + 25}{30} = \frac{34}{30} = \frac{17}{15}$

11. Given the transformation $u = 2x - 3y$ and $v = -x + y$.

(3 points) a. Rewrite the transformation equations with x and y in terms of u and v.

$y = x + v \Rightarrow u = 2x - 3(x+v)$

$u = 2x - 3x - 3v$

$u = -x - 3v$

$x = -u - 3v \Rightarrow y - u - 3v + v$

$y - u - 2v$

(4 points) b. Use the transformation to transform the region R in the xy-plane bounded by the lines $x = -3$, $x = 0$, $y = x$, and $y = x + 1$ to the uv-plane.

$x = -3 \Rightarrow -u - 3v - 3 \Rightarrow u + 3v = 3$

$x = 0 \Rightarrow -u - 3v = 0 \Rightarrow u + 3v = 0$

$y = x \Rightarrow -u - 2v = -u - 3v \Rightarrow v = 0$

$y = x + 1 \Rightarrow -u - 2v = -u - 3v + 1 \Rightarrow v = 1$

(3 points) c. Find the Jacobian, $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1$$

or 1

(7 points) 12. Given $\vec{F} = \left(4x^3y^2 - 2xy^3\right)\hat{i} + \left(2x^4y - 3x^2y^2 + 4y^3\right)\hat{j}$. Verify that $\vec{F}$ is conservative. Then, find the potential function, $f(x,y)$.

P Q

$$\frac{\partial P}{\partial y} = 8x^3y - 6xy^2 = \frac{\partial Q}{\partial x} = 8x^3y - 6xy^2$$

$\therefore$ $\vec{F}$ is conservative

$$f(x,y) = \int P\,dx = \int (4x^3y^2 - 2xy^3)\,dx$$

$$= x^4y^2 - x^2y^3 + g(y)$$

$$f_y = 2x^4y - 3x^2y^2 + g'(y) = 2x^4y - 3x^2y^2 + 4y^3 = Q$$

$$g'(y) = 4y^3$$

$$g(y) = \int 4y^3\,dy = y^4 + C$$

$$f(x,y) = x^4y^2 - x^2y^3 + y^4 + C$$