MATH 280 – EXAM #3
Summer Session 2019

Name: Key

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes or the textbook. You may not use any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam. You may not use a graphing calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(9 points) *1.* Find the maximum and minimum values of $f(x,y) = 3x - y + 6$ on the circle $x^2 + y^2 = 4$

$\vec{\nabla} f = 3\hat{\imath} - \hat{\jmath}$ $\quad \vec{\nabla} g = 2x\hat{\imath} + 2y\hat{\jmath}$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$3 = 2x\lambda \quad -1 = 2y\lambda$

$\frac{3}{2\lambda} = x \quad \frac{-1}{2\lambda} = y$

$\left(\frac{3}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 = 4$

$\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 4$

$10 = 16\lambda^2$

$\frac{10}{16} = \lambda^2 \quad \lambda = \frac{\pm\sqrt{10}}{4}$

$\lambda = \frac{\sqrt{10}}{4} \quad x = \frac{6}{\sqrt{10}} \quad y = -\frac{2}{\sqrt{10}}$

$\lambda = -\frac{\sqrt{10}}{4} \quad x = -\frac{6}{\sqrt{10}} \quad y = \frac{2}{\sqrt{10}}$

$f\left(\frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = 3\left(\frac{6}{\sqrt{10}}\right) - \left(-\frac{2}{\sqrt{10}}\right) + 6$

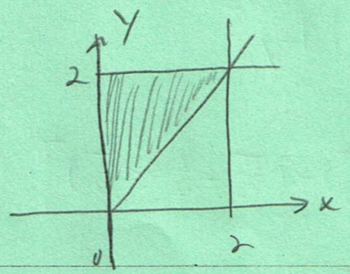
$= \frac{20}{\sqrt{10}} + 6$ maximum

$f\left(-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) = 3\left(-\frac{6}{\sqrt{10}}\right) - \left(\frac{2}{\sqrt{10}}\right) + 6$

$= -\frac{20}{\sqrt{10}} + 6$ minimum

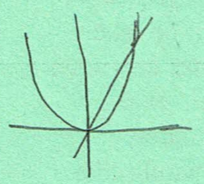
(4 points) *2.* Sketch the region of integration and reverse the order of integration.

$$\int_0^2 \int_x^2 2y^2 \sin(xy)\, dy\, dx$$



$$\int_0^2 \int_0^y 2y^2 \sin(xy)\, dx\, dy$$

(9 points) 3. Evaluate the double integral $\iint_D (x+2y)\,dA$ where D is the region bounded by $y=2x$ and $y=x^2$.



$$\int_0^2\int_{x^2}^{2x}(x+2y)\,dy\,dx = \int_0^2 (xy+y^2)\Big|_{x^2}^{2x}\,dx$$

$$= \int_0^2 \left[\left[x(2x)+(2x)^2\right]-\left[x(x^2)+(x^2)^2\right]\right]dx$$

$$= \int_0^2 (2x^2+4x^2-x^3-x^4)\,dx = \int_0^2 (6x^2-x^3-x^4)\,dx$$

$$= \left(2x^3-\frac{x^4}{4}-\frac{x^5}{5}\right)\Big|_0^2 = 16-4-\frac{32}{5} = 12-\frac{32}{5} = \frac{60-32}{5} = \frac{28}{5}$$

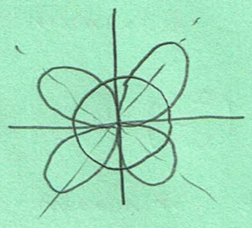
$x^2=2x$

$x^2-2x=0$

$x(x-2)=0$

$x=0,2$

(9 points) 4. Use a double integral to find the area of the region inside one leaf of the graph $r=2\sin 2\theta$ and outside $r=1$.



$$\int_{\pi/12}^{5\pi/12}\int_1^{2\sin 2\theta} r\,dr\,d\theta = \int_{\pi/12}^{5\pi/12} \frac{r^2}{2}\Big|_1^{2\sin 2\theta}\,d\theta = \frac{1}{2}\int_{\pi/12}^{5\pi/12}(4\sin^2 2\theta-1)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/12}^{5\pi/12}\left[4\left(\frac{1-\cos 4\theta}{2}\right)-1\right]d\theta = \frac{1}{2}\int_{\pi/12}^{5\pi/12}(1-2\cos 4\theta)\,d\theta$$

$2\sin 2\theta = 1$

$\sin 2\theta = 1/2$

$2\theta = \pi/6 \quad 2\theta = 5\pi/6$

$\theta = \pi/12 \quad \theta = 5\pi/12$

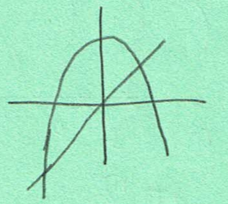
$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin 4\theta\right)\Big|_{\pi/12}^{5\pi/12}$$

$$= \frac{1}{2}\left[\left(\frac{5\pi}{12}-\frac{1}{2}\sin\frac{5\pi}{3}\right)-\left(\frac{\pi}{12}-\frac{1}{2}\sin\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\right]$$

$$= \frac{\pi}{6}+\frac{\sqrt{3}}{4}$$

(9 points) 5. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 2 - x^2$ and the line $y = x$, while the top of the solid is bounded by the paraboloid $z = x^2$.



$$2 - x^2 = x$$
$$0 = x^2 + x - 2$$
$$0 = (x+2)(x-1)$$
$$x = -2, 1$$

$$\int_{-2}^{1}\int_{x}^{2-x^2}\int_{0}^{x^2} dz\,dy\,dx = \int_{-2}^{1}\int_{x}^{2-x^2} z\Big|_0^{x^2} dy\,dx$$

$$= \int_{-2}^{1}\int_{x}^{2-x^2} x^2\,dy\,dx = \int_{-2}^{1} x^2 y\Big|_x^{2-x^2} dx = \int_{-2}^{1} x^2[2-x^2-x]\,dx$$

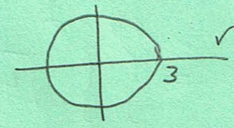
$$= \int_{-2}^{1}(2x^2 - x^4 - x^3)\,dx = \left(\frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right)\Big|_{-2}^{1}$$

$$= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(-\frac{16}{3} + \frac{32}{5} - 4\right) = \frac{2}{3} - \frac{1}{5} - \frac{1}{4} + \frac{16}{3} - \frac{32}{5} + 4$$

$$= \frac{18}{3} - \frac{33}{5} - \frac{1}{4} + 4 = 10 - \frac{33}{5} - \frac{1}{4} = \frac{200 - 132 - 5}{20} = \frac{63}{20}$$

(9 points) 6. Use cylindrical coordinates to find the volume of the solid that is bounded below by the paraboloid $z = x^2 + y^2$ and above by $z = 9$.

$$x^2 + y^2 = 9$$



$r = 3$

$$\int_0^{2\pi}\int_0^3\int_{r^2}^{9} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 rz\Big|_{r^2}^{9} dr\,d\theta = \int_0^{2\pi}\int_0^3 r(9-r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (9r - r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{9}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^3 d\theta$$

$$= \int_0^{2\pi}\left(\frac{9}{2}(3)^2 - \frac{1}{4}(3)^4\right) d\theta = \int_0^{2\pi}\left(\frac{81}{2} - \frac{81}{4}\right) d\theta = \int_0^{2\pi}\frac{162-81}{4}\,d\theta$$

$$= \int_0^{2\pi}\frac{81}{4}\,d\theta = \frac{81}{4}\theta\Big|_0^{2\pi} = \frac{81\pi}{2}$$

(9 points) 7. Find the volume of the solid bounded below by the *xy*-plane, on the sides by the sphere $\rho = 2$ and above by the cone $\varphi = \pi/3$.

$$\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \frac{\rho^3}{3}\sin\varphi \Big|_0^2 \, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \frac{8}{3}\sin\varphi \, d\varphi\, d\theta = \frac{8}{3}\int_0^{2\pi} (-\cos\varphi)\Big|_{\pi/3}^{\pi/2} d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{4}{3}\theta\Big|_0^{2\pi} = \frac{8\pi}{3}$$

(9 points) 8. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.

$$x^2 + y^2 = 4$$

$$\iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\, dA = \int_0^{2\pi}\int_0^{2} \sqrt{4r^2+1}\, r\, dr\, d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \frac{2}{3}(4r^2+1)^{3/2}\Big|_0^2 d\theta = \frac{1}{12}\int_0^{2\pi} (17^{3/2} - 1)\, d\theta$$

$$= \frac{17^{3/2}-1}{12}\,\theta\Big|_0^{2\pi} = \frac{\pi}{6}(17^{3/2} - 1)$$

(8 points) *9.* Given $\vec{F} = (x+y^2)\hat{i} + (xz)\hat{j} + (y+z)\hat{k}$ and the curve $\vec{r}(t) = t^2\hat{i} + t^3\hat{j} - 2t\hat{k}$, $0 \le t \le 2$. Find the work done by $\vec{F}$ over $\vec{r}(t)$.

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 3t^2\hat{j} - 2\hat{k}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = (x+y^2)2t + (xz)3t^2 + (y+z)(-2)$$

$$= (t^2+t^6)2t + (t^2)(-2t)3t^2 + (t^3-2t)(-2)$$

$$= 2t^3 + 2t^7 - 6t^5 - 2t^3 + 4t$$

$$= 2t^7 - 6t^5 + 4t$$

$$\int_0^2 (2t^7 - 6t^5 + 4t)\,dt = \left(\frac{2t^8}{8} - t^6 + 2t^2\right)\Big|_0^2$$

$$= 64 - 64 + 8 = 8$$

(8 points) 10. Evaluate $\int_C (2x-3y+z)\,ds$ where C is the straight line segment from $(1,2,-3)$ to $(-2,1,3)$

$(1,2,-3)$ to $(-2,1,3)$ $\quad \vec{v} = \langle -3, -1, 6 \rangle$

$$x = 1-3t$$
$$y = 2-t \quad , 0 \le t \le 1$$
$$z = -3+6t$$

$$\int_C (2x-3y+z)\,ds = \int_0^1 [2(1-3t) - 3(2-t) + (-3+6t)]\sqrt{(-3)^2+(-1)^2+(6)^2}\,dt$$

$$= \int_0^1 (2-6t-6+3t-3+6t)\sqrt{9+1+36}\,dt = \int_0^1 (3t-7)\sqrt{46}\,dt$$

$$= \sqrt{46}\left(\frac{3t^2}{2} - 7t\right)\Big|_0^1 = \sqrt{46}\left(\frac{3}{2} - 7\right) = \frac{-11\sqrt{46}}{2}$$

11. Given the transformation $u = 2x - 3y$, $v = -x + y$.

(3 points) a. Solve the system for *x* and *y* in terms of *u* and *v*.

$u = 2x - 3y \qquad v = -x + y$

$v + x = y$

$u = 2x - 3(v + x) \qquad y = v - u - 3v$

$u = 2x - 3v - 3x \qquad y = -u - 2v$

$u + 3v = -x$

$x = -u - 3v$

(3 points) b. Find the Jacobian, $\dfrac{\partial(x, y)}{\partial(u, v)}$.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1$$

(5 points) c. Find the image under the given transformation of the triangular region bounded by the lines $x = -3$, $x = 0$, $y = x$, and $y = x + 1$.

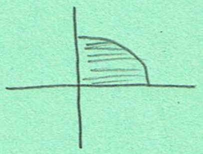
$x = -3 \Rightarrow -u - 3v = -3 \Rightarrow -u = 3v - 3 \Rightarrow u = -3v + 3$

$x = 0 \Rightarrow -u - 3v = 0 \Rightarrow u = -3v$

$y = x \Rightarrow -u - 2v = -u - 3v \Rightarrow -2v = -3v \Rightarrow v = 0$

$y = x + 1 \Rightarrow -u - 2v = -u - 3v + 1 \Rightarrow -2v = -3v + 1 \Rightarrow v = 1$

12. Given a thin plate of density $\rho(x, y) = y$ bounded by the parabola $x = 1 - y^2$ and the coordinate axes in first quadrant.



(2 points) a. Set up the integral that would compute M_y.

$$\int_0^1 \int_0^{1-y^2} y\,x\,dx\,dy$$

(2 points) b. Set up the integral that would compute I_x.

$$\int_0^1 \int_0^{1-y^2} y\,y^2\,dx\,dy = \int_0^1 \int_0^{1-y^2} y^3\,dx\,dy$$

(4 points) 13. Given that the vector field $\vec{F}(x, y, z) = 2xyz^2\hat{\imath} + x^2z^2\hat{\jmath} + 2x^2yz\hat{k}$ is a gradient field with $f(x, y, z) = x^2yz^2$. Find the work done by $\vec{F}$ in moving a particle from $(2,1,-4)$ to $(3,-2,1)$.

$$\int_C \vec{F} \cdot d\vec{r} = f(3,-2,1) - f(2,1,-4)$$

$$= (9)(-2)(1)^2 - (2)^2(1)(-4)^2$$

$$= -18 - 64$$

$$= -82$$