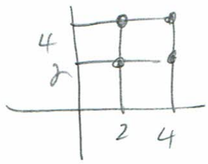
Math 280 Exam #3

1. $\iint_R (x^2y+2y)\,dA \quad R: [0,4]\times[0,4]$

4/

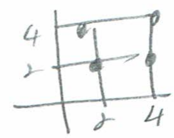


$\Delta A = 4$

$A \approx f(2,2)\cdot 4 + f(4,2)\cdot 4 + f(2,4)\cdot 4 + f(4,4)\cdot 4$

$= (12+36+24+72)\,4 = 576$

$\iint_R (xy^2+2x)\,dA \quad R: [0,4]\times[0,4]$



$\Delta A = 4$

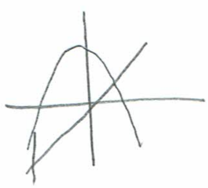
$A \approx f(2,2)\cdot 4 + f(4,2)\cdot 4 + f(2,4)\cdot 4 + f(4,4)\cdot 4$

$= (12+24+36+72)\cdot 4 = 576$

2. $\iint_D (x^2+2y)\,dA$

D: $y = 2-x^2$ and $y = x$

4/



$2-x^2 = x$

$0 = x^2+x-2$

$0 = (x+2)(x-1)$

$x = -2, 1$

$\int_{-2}^{1}\int_{x}^{2-x^2} (x^2+2y)\,dy\,dx = \int_{-2}^{1} (x^2y+y^2)\Big|_{x}^{2-x^2} dx$

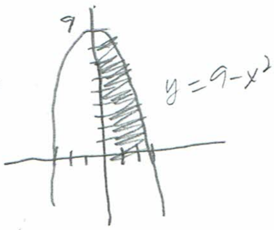
$= \int_{-2}^{1} \left[\left(x^2(2-x^2)+(2-x^2)^2\right) - (x^3+x^2)\right] dx$

$= \int_{-2}^{1} (2x^2 - x^4 + 4 - 4x^2 + x^4 - x^3 - x^2)\,dx = \int_{-2}^{1} (-x^3-3x^2+4)\,dx$

$= \int_{-2}^{1} \left(-\frac{x^4}{4} - x^3 + 4x\right)\Big|_{-2}^{1} = \left(-\frac{1}{4} - 1 + 4\right) - \left(-\frac{(-2)^4}{4} - (-2)^3 + 4(-2)\right) = \frac{27}{4}$

3. $\int_0^3 \int_0^{9-x^2} \frac{xe^{3y}}{9-y}\,dy\,dx$

6/



$\int_0^9 \int_0^{\sqrt{9-y}} \frac{xe^{3y}}{9-y}\,dx\,dy$

$x^2 = 9-y$

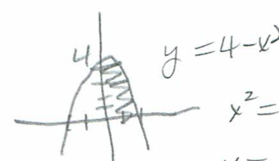
$x = \sqrt{9-y}$

$= \int_0^9 \frac{x^2}{2}\,\frac{e^{3y}}{9-y}\Big|_0^{\sqrt{9-y}} dy = \frac{1}{2}\int_0^9 9-y\left(\frac{e^{3y}}{9-y}\right) dy$

$= \frac{1}{2}\int_0^9 e^{3y}\,dy = \frac{1}{2}\,\frac{e^{3y}}{3}\Big|_0^9$

$= \frac{1}{6}\left(e^{27}-1\right)$

$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$



$x^2 = 4-y$

$x = \sqrt{4-y}$

$\int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dx\,dy = \int_0^4 \frac{x^2}{2}\,\frac{e^{2y}}{4-y}\Big|_0^{\sqrt{4-y}} dy$

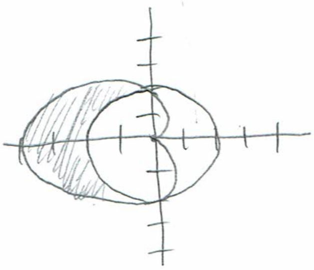
$= \frac{1}{2}\int_0^4 4-y\left(\frac{e^{2y}}{4-y}\right) dy$

$= \frac{1}{2}\int_0^4 e^{2y}\,dy = \frac{1}{2}\,\frac{e^{2y}}{2}\Big|_0^4$

$= \frac{1}{4}\left(e^4-1\right)$

④ $r = 2 - 2\cos\theta$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	0	2	4	2	0



$2 - 2\cos\theta = 2$

$\cos\theta = 0$

$\theta = \pi/2,\ 3\pi/2$

$$\int_{\pi/2}^{3\pi/2}\int_{2}^{2-2\cos\theta} r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2} \frac{r^2}{2}\Big|_{2}^{2-2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left[(2-2\cos\theta)^2 - 2^2\right] d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{3\pi/2} (4 - 8\cos\theta + 4\cos^2\theta - 4)\, d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left[-8\cos\theta + 4\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{3\pi/2} (-8\cos\theta + 2 + 2\cos 2\theta)\, d\theta$$

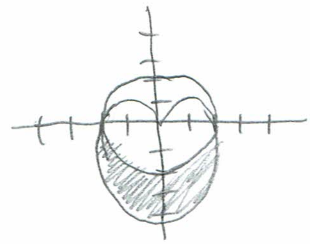
$$= \frac{1}{2}\left[-8\sin\theta + 2\theta + \sin 2\theta\right]\Big|_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{2}\left[(8 + 3\pi) - (-8 + \pi)\right]$$

$$= \frac{1}{2}(16 + 2\pi) = 8 + \pi$$

$r = 2 - 2\sin\theta$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	2	0	2	4	2



$2 - 2\sin\theta = 2$

$\sin\theta = 0$

$\theta = \pi,\ 2\pi$

$$\int_{\pi}^{2\pi}\int_{2}^{2-2\sin\theta} r\,dr\,d\theta = \int_{\pi}^{2\pi} \frac{r^2}{2}\Big|_{2}^{2-2\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} \left[(2-2\sin\theta)^2 - 2^2\right] d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} \left[4 - 8\sin\theta + 4\sin^2\theta - 4\right] d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} \left[-8\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} (-8\sin\theta + 2 - 2\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(8\cos\theta + 2\theta - \sin 2\theta\right)\Big|_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left[(8 + 4\pi) - (-8 + 2\pi)\right]$$

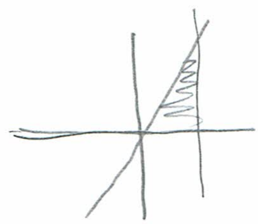
$$= \frac{1}{2}(16 + 2\pi) = 8 + \pi$$

⑤ $2y+4z-x^2=5$

$4z=x^2-2y+5$

$z=\dfrac{x^2-2y+5}{4}$

$f_x=\frac{1}{2}x \quad f_y=-\frac{1}{2}$



7/ $\int_0^2\int_0^{2x}\sqrt{1+(\frac{1}{2}x)^2+(\frac{1}{2})^2}\,dy\,dx=\int_0^2\int_0^{2x}\sqrt{\frac{5}{4}+\frac{1}{4}x^2}\,dy\,dx=\frac{1}{2}\int_0^2\int_0^{2x}\sqrt{5+x^2}\,dy\,dx$

$=\frac{1}{2}\int_0^2 y\sqrt{5+x^2}\Big|_0^{2x}dx=\frac{1}{2}\int_0^2 2x\sqrt{5+x^2}\,dx=\frac{1}{2}\int_0^2 u^{1/2}du=\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_0^2$

$u=5+x^2$
$du=2x\,dx$

$=\frac{1}{3}(5+x^2)^{3/2}\Big|_0^2=\frac{1}{3}\left[9^{3/2}-5^{3/2}\right]=\frac{1}{3}(27-5^{3/2})$

⑥ $y=x+2,\ y=x^2$

$x^2=x+2$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2,-1$

a/ $m=\int_{-1}^{2}\int_{x^2}^{x+2}kx^2\,dy\,dx$

a/ $M_y=\int_{-1}^{2}\int_{x^2}^{x+2}x(kx^2)\,dy\,dx=\int_{-1}^{2}\int_{x^2}^{x+2}kx^3\,dy\,dx$

a/ $I_x=\int_{-1}^{2}\int_{x^2}^{x+2}y^2(kx^2)\,dy\,dx=\int_{-1}^{2}\int_{x^2}^{x+2}kx^2y^2\,dy\,dx$

$y=x+6,\ y=x^2$

$x^2=x+6$

$x^2-x-6=0$

$(x-3)(x+2)=0$

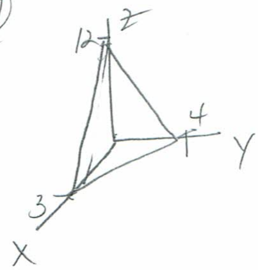
$x=3,-2$

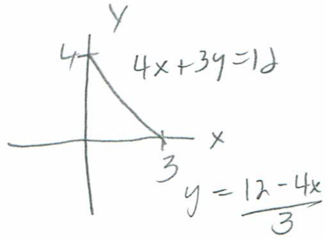
$M=\int_{-2}^{3}\int_{x^2}^{x+6}ky^2\,dy\,dx$

$M_x=\int_{-2}^{3}\int_{x^2}^{x+6}y(ky^2)\,dy\,dx=\int_{-2}^{3}\int_{x^2}^{x+6}ky^3\,dy\,dx$

$I_y=\int_{-2}^{3}\int_{x^2}^{x+6}x^2(ky^2)\,dy\,dx=\int_{-2}^{3}\int_{x^2}^{x+6}kx^2y^2\,dy\,dx$

(7)





$$\int_0^3\int_0^{4-\frac{4}{3}x}\int_0^{12-3y-4x} dz\,dy\,dx$$

$$=\int_0^3\int_0^{4-\frac{4}{3}x} z\Big|_0^{12-3y-4x} dy\,dx = \int_0^3\int_0^{4-\frac{4}{3}x}(12-3y-4x)\,dy\,dx$$

$$=\int_0^3\left(12y-\frac{3}{2}y^2-4xy\right)\Big|_0^{4-\frac{4}{3}x}dx$$

$$=\int_0^3\left[12\left(4-\frac{4}{3}x\right)-\frac{3}{2}\left(4-\frac{4}{3}x\right)^2-4x\left(4-\frac{4}{3}x\right)\right]dx$$

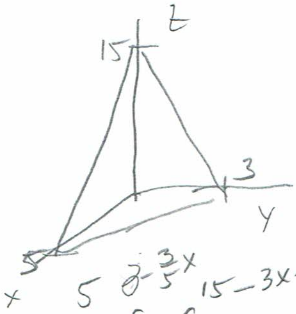
$$=\int_0^3\left(48-16x-\frac{3}{2}\left(16-\frac{32}{3}x+\frac{16}{9}x^2\right)-16x+\frac{16}{3}x^2\right)dx$$

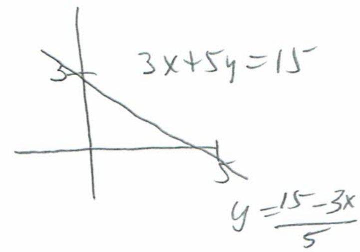
$$=\int_0^3\left(48-16x-24+16x-\frac{8}{3}x^2-16x+\frac{16}{3}x^2\right)dx$$

$$=\int_0^3\left(\frac{8}{3}x^2-16x+24\right)dx$$

$$=\left(\frac{8}{9}x^3-8x^2+24x\right)\Big|_0^3$$

$$=24-72+72=24$$





$$\int_0^5\int_0^{3-\frac{3}{5}x}\int_0^{15-3x-5y} dz\,dy\,dx$$

$$=\int_0^5\int_0^{3-\frac{3}{5}x} z\Big|_0^{15-3x-5y} dy\,dx=\int_0^5\int_0^{3-\frac{3}{5}x}(15-3x-5y)\,dy\,dx$$

$$=\int_0^5\left(15y-3xy-\frac{5}{2}y^2\right)\Big|_0^{3-\frac{3}{5}x}dx$$

$$=\int_0^5\left[15\left(3-\frac{3}{5}x\right)-3x\left(3-\frac{3}{5}x\right)-\frac{5}{2}\left(3-\frac{3}{5}x\right)^2\right]dx$$

$$=\int_0^5\left[45-9x-9x+\frac{9x^2}{5}-\frac{5}{2}\left(9-\frac{18}{5}x+\frac{9}{25}x^2\right)\right]dx$$

$$=\int_0^5\left(45-9x-9x+\frac{9}{5}x^2-\frac{45}{2}+9x-\frac{9}{10}x^2\right)dx$$

$$=\int_0^5\left(\frac{9}{10}x^2-9x+\frac{45}{2}\right)dx$$

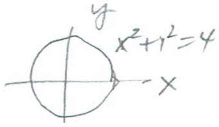
$$=\left(\frac{3}{10}x^3-\frac{9}{2}x^2+\frac{45}{2}x\right)\Big|_0^5$$

$$=\frac{3}{10}(125)-\frac{9}{2}(25)+\frac{45}{2}(5)$$

$$=\frac{75}{2}-\frac{225}{2}+\frac{225}{2}=\frac{75}{2}$$

(8)

$$\iiint_E yz\,dV$$



$$\int_0^{2\pi}\int_0^{2}\int_0^{y} r\sin\theta\, z\, r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2}\int_0^{r\sin\theta} r^2 z\sin\theta\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2} r^2\frac{z^2}{2}\sin\theta\Big|_0^{r\sin\theta}dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^{2} r^2\sin\theta(r\sin\theta)^2\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^{2} r^4\sin^3\theta\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\frac{r^5}{5}\sin^3\theta\Big|_0^{2}d\theta$$

$$=\frac{32}{10}\int_0^{2\pi}\sin^3\theta\,d\theta=\frac{16}{5}\int_0^{2\pi}\sin^2\theta\sin\theta\,d\theta$$

$$=\frac{16}{5}\int_0^{2\pi}(1-\cos^2\theta)\sin\theta\,d\theta$$

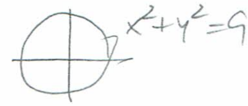
$$=\frac{16}{5}\int_0^{2\pi}(\sin\theta-\cos^2\theta\sin\theta)\,d\theta$$

$$=\frac{16}{5}\left[-\cos\theta+\frac{\cos^3\theta}{3}\right]\Big|_0^{2\pi}$$

$$=\frac{16}{5}\left[\left(-1+\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$=\frac{16}{5}\left(-\frac{2}{3}+\frac{2}{3}\right)=0$$

$$\iiint_E xz\,dV$$



$$\int_0^{2\pi}\int_0^{3}\int_0^{x} r\cos\theta\, z\, r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{3}\int_0^{r\cos\theta} r^2 z\cos\theta\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{3} r^2\frac{z^2}{2}\cos\theta\Big|_0^{r\cos\theta}dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^{3} r^2(r\cos\theta)^2\cos\theta\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^{3} r^4\cos^3\theta\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\frac{r^5}{5}\cos^3\theta\Big|_0^{3}d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\frac{243}{5}\cos^3\theta\,d\theta=\frac{243}{10}\int_0^{2\pi}\cos^2\theta\cos\theta\,d\theta$$

$$=\frac{243}{10}\int_0^{2\pi}(1-\sin^2\theta)\cos\theta\,d\theta$$

$$=\frac{243}{10}\int_0^{2\pi}(\cos\theta-\sin^2\theta\cos\theta)\,d\theta$$

$$=\frac{243}{10}\left[\sin\theta-\frac{\sin^3\theta}{3}\right]\Big|_0^{2\pi}$$

$$=\frac{243}{10}[0]=0$$

a

$$\iiint_E x e^{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)e^{\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$x^2+y^2+z^2 \le 1$ first octant

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3\sin^2\phi\cos\theta\, e^{\rho^2}\,d\rho\,d\phi\,d\theta$$

$$\int \rho^3 e^{\rho^2}\,d\rho$$

$u=\rho^2$ $\quad dv=\rho e^{\rho^2}d\rho$

$du=2\rho\,d\rho$ $\quad v=\frac{1}{2}e^{\rho^2}$ $\quad w=\rho^2$ $\quad dw=2\rho\,d\rho$

$$\frac{1}{2}\rho^2 e^{\rho^2} - \int 2\rho\left(\frac{1}{2}e^{\rho^2}\right)d\rho$$

$$\frac{1}{2}\rho^2 e^{\rho^2} - \int \rho e^{\rho^2}d\rho$$

$$\frac{1}{2}\rho^2 e^{\rho^2} - \frac{1}{2}e^{\rho^2}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{2}\rho^2 e^{\rho^2} - \frac{1}{2}e^{\rho^2}\right)\sin^2\phi\cos\theta\Big|_0^1 d\phi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\left(\frac{1}{2}e - \frac{1}{2}e\right) - \left(0-\frac{1}{2}\right)\right]\sin^2\phi\cos\theta\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{2}\sin^2\phi\cos\theta\,d\phi\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1-\cos 2\phi}{2}\right)\cos\theta\,d\phi\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(\phi - \frac{\sin 2\phi}{2}\right)\Big|_0^{\pi/2} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\frac{\pi}{2}\cos\theta\,d\theta$$

$$= \frac{\pi}{8}\int_0^{\pi/2}\cos\theta\,d\theta$$

$$= \frac{\pi}{8}\sin\theta\Big|_0^{\pi/2} = \frac{\pi}{8}$$

(10) $z = x^2+y^2 \qquad z = 2-x^2-y^2 \longrightarrow$

$x^2+y^2 = 2-x^2-y^2$

$2x^2+2y^2=2$

$x^2+y^2=1 \rightarrow r^2=1 \rightarrow r=1$

$$\iiint_E dz\,dy\,dx \xrightarrow{\text{polar}} \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 rz\Big|_{r^2}^{2-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\left[2-r^2-r^2\right] dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2r-2r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(r^2-\tfrac{1}{2}r^4\right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}\,d\theta = \tfrac{1}{2}\theta\Big|_0^{2\pi} = \pi$$

7/

(11) $\displaystyle\iint \frac{x-y}{x+y}\,dA$ $\qquad y=x,\ y=x+2,\ x+y=2,\ x+y=4$

$u = x-y,\quad v = x+y$

3/a. $x = y+u \qquad v = y+u+y$

$v = 2y+u$

$2y = -u+v$

$x = \left(-\frac{1}{2}u+\frac{1}{2}v\right)+u \qquad y = -\frac{1}{2}u+\frac{1}{2}v$

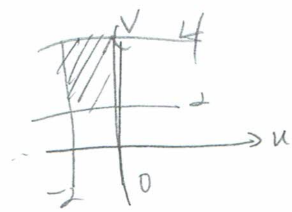
$x = \frac{1}{2}u+\frac{1}{2}v$

3/b. $y=x:\quad -\frac{1}{2}u+\frac{1}{2}v = \frac{1}{2}u+\frac{1}{2}v \Rightarrow 0=u$

$y=x+2 \quad -\frac{1}{2}u+\frac{1}{2}v = \frac{1}{2}u+\frac{1}{2}v+2 \Rightarrow -2=u$

$x+y=2 \quad \left(\frac{1}{2}u+\frac{1}{2}v\right)+\left(-\frac{1}{2}u+\frac{1}{2}v\right)=2 \Rightarrow v=2$

$x+y=4 \quad \left(\frac{1}{2}u+\frac{1}{2}v\right)+\left(-\frac{1}{2}u+\frac{1}{2}v\right)=4 \Rightarrow v=4$



2/c. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = 1/4 + 1/4 = 1/2$

2/d. $\displaystyle\int_{-2}^{0}\int_{2}^{4} \frac{u}{v}\,dv\,du$