MATH 280 – EXAM #3 **Name:** KEY

Fall Semester 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes or the textbook. You may not use any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam. You may not use a graphing calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(8 points) 1. Given $f(x,y) = 2 - x^4 + 2x^2 - y^2$. Find all local maxima, local minima, and saddle points.

$f_x = -4x^3 + 4x$ $\qquad f_y = -2y$

$f_x = -4x^3 + 4x = 0$ $\qquad f_y = -2y = 0$

$y = 0$

$-4x(x^2-1) = 0$

$x = 0, \pm 1$

$(0,0)$ $(1,0)$ $(-1,0)$

$f_{xx} = -12x^2 + 4$ $\quad f_{xy} = 0$ $\quad f_{yy} = -2$

$(0,0)$: $D = (4)(-2) - (0)^2 = -8 < 0$ saddle point

$(1,0)$: $D = (-8)(-2) - (0)^2 = 16 > 0$ $f_{xx} = -8 > 0$ local maximum

$(-1,0)$: $D = (-8)(-2) - (0)^2 = 16 > 0$ $f_{xx} = -8 > 0$ local maximum

(8 points) 2. Find the minimum and maximum values of $f(x,y) = 3x + y$ subject to the constraint $x^2 + y^2 = 10$.

$\vec{\nabla} f = 3\hat{i} + \hat{j}$ $\quad$ $\vec{\nabla} g = 2x\hat{i} + 2y\hat{j}$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$3 = 2x\lambda$ $\quad$ $1 = 2y\lambda$ $\quad$ $x^2 + y^2 = 10$ $\quad$ $\lambda = \frac{1}{2}$ $\quad$ $x = 3$ $\quad$ $y = 1$

$x = \frac{3}{2\lambda}$ $\quad$ $y = \frac{1}{2\lambda}$ $\quad$ $x^2 + y^2 = 10$ $\quad$ $\lambda = -\frac{1}{2}$ $\quad$ $x = -3$ $\quad$ $y = -1$

$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 10$ $\quad$ $f(3,1) = 10$ maximum

$\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10$ $\quad$ $f(-3,-1) = -10$ minimum

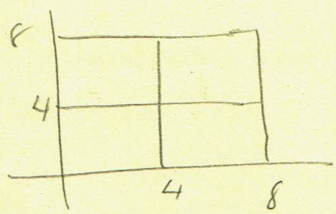
$\frac{10}{4\lambda^2} = 10$

$10 = 40\lambda^2$

$\frac{1}{4} = \lambda^2$

$\lambda = \pm\frac{1}{2}$

(5 points) 3. Estimate $\iint_R (x + xy^2)\,dA$ where $R = [0,8]\times[0,8]$ by dividing R into four equal squares and evaluating the function at the upper right corner of each square.



$\Delta A = 16$

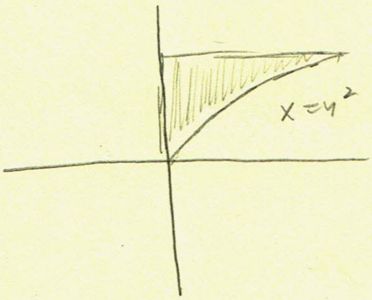
$A \approx f(4,4)\cdot 16 + f(4,8)\cdot 16 + f(8,4)\cdot 16 + f(8,8)\cdot 16$

$A \approx 68\cdot 16 + 260\cdot 16 + 136\cdot 16 + 520\cdot 16$

$A \approx 15744$

(4 points) 4. Sketch the region of integration and reverse the order of integration.

$$\int_0^4\int_{\sqrt{x}}^2 \frac{dydx}{y^3+1}$$



$$\int_0^2\int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy$$

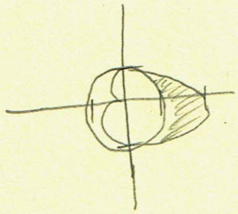
(7 points) 5. Evaluate the double integral $\iint_D (x^2+2y)\,dA$ where D is the region bounded by $y=x$, $y=x^3$, $x\geq 0$.

$$\int_0^1\int_{x^3}^x (x^2+2y)\,dydx = \int_0^1 (x^2y+y^2)\Big|_{x^3}^x dx = \int_0^1 \left[(x^3+x^2)-(x^5+x^6)\right] dx$$

$$=\int_0^1 (x^3+x^2-x^5-x^6)\,dx = \left(\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^6}{6}-\frac{x^7}{7}\right)\Big|_0^1$$

$$=\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{7} = \frac{21+28-14-12}{84} = \frac{23}{84}$$

(7 points) 6. Use a double integral to find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.



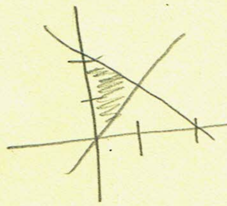
$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^2}{2}\Big|_{1}^{1+\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} [(1+\cos\theta)^2 - 1]\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1 + 2\cos\theta + \cos^2\theta - 1)\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (2\cos\theta + \cos^2\theta)\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left(2\sin\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right)\Big|_{-\pi/2}^{\pi/2} = \frac{1}{2}\left[\left(2+\frac{\pi}{4}\right) - \left(-2-\frac{\pi}{4}\right)\right] = \frac{1}{2}\left(4 + \frac{\pi}{2}\right) = 2 + \frac{\pi}{4}$$

(7 points) 7. Find the volume of the solid whose base is the region in the *xy*-plane that is bounded by the lines $y = x$, $x = 0$, and $x + y = 2$ while the top of the solid is bounded by the paraboloid $z = x^2 + y^2$.



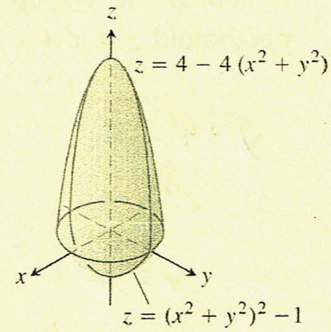
$$\int_0^1\int_x^{2-x}\int_0^{x^2+y^2} dz\,dy\,dx = \int_0^1\int_y^{2-x} z\Big|_0^{x^2+y^2} dy\,dx$$

$$= \int_0^1 \left(x^2y + \frac{y^3}{3}\right)\Big|_x^{2-x} dx = \int_0^1 \left[\left[x^2(2-x) + \frac{(2-x)^3}{3}\right] - \left[x^3 + \frac{x^3}{3}\right]\right] dx$$

$$= \int_0^1 \left(2x^2 - x^3 + \frac{(2-x)^3}{3} - \frac{4}{3}x^3\right) dx = \int_0^1 \left[2x^2 - \frac{7}{3}x^3 + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2}{3}x^3 - \frac{7}{12}x^4 - \frac{(2-x)^4}{12}\right]\Big|_0^1$$

$$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(-\frac{16}{12}\right) = \frac{2}{3} - \frac{7}{12} - \frac{1}{12} + \frac{4}{3} = 2 - \frac{8}{12} = 2 - \frac{2}{3} = \frac{4}{3}$$

(7 points) *8.* Find the volume of the solid bounded above by $z = 4 - 4(x^2 + y^2)$ and bounded below by $z = (x^2 + y^2)^2 - 1$. Use cylindrical coordinates.



$$\int_0^{2\pi} \int_0^1 \int_{(r^2)^2-1}^{4-4r^2} r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 rz \Big|_{r^4-1}^{4-4r^2} dr\, \theta$$

$$= \int_0^{2\pi} \int_0^1 r\left[(4-4r^2) - (r^4-1)\right] dr\, d\theta = \int_0^{2\pi} \int_0^1 r\,(5-4r^2-r^4)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (5r-4r^3-r^5)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{5}{2}r^2 - r^4 - \frac{1}{6}r^6\right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{5}{2} - 1 - \frac{1}{6}\right) d\theta = \int_0^{2\pi} \frac{15-6-1}{6}\, d\theta = \int_0^{2\pi} \frac{8}{6}\, d\theta = \frac{4}{3}\theta \Big|_0^{2\pi} = \frac{8\pi}{3}$$

(5 points) *9.* Set up the integral in spherical coordinates that would find the volume of the solid that lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

$$\int_0^{2\pi} \int_0^{\pi} \int_2^{3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

(5 points) *10.* Set up the integral that would compute the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.

$x^2+y^2=4$

$r^2=4$

$r=2$

$$\iint \sqrt{(2x)^2+(2y)^2+1}\, dA$$

$$\int_0^{2\pi}\int_0^{2} \sqrt{4r^2+1}\; r\,dr\,d\theta$$

(7 points) *11.* Given $\vec{F} = (x+y^2)\hat{i} + (xz)\hat{j} + (y+z)\hat{k}$ and the curve $\vec{r}(t) = t^2\hat{i} + t^3\hat{j} - 2t\hat{k}$, $0 \le t \le 2$. Find the work done by $\vec{F}$ over $\vec{r}(t)$.

$\vec{r}'(t) = 2t\hat{i} + 3t^2\hat{j} - 2\hat{k}$

$\vec{F}\cdot\vec{r}'(t) = [(t^2+t^6)\hat{i} + (-2t^3)\hat{j} + (t^3-2t)\hat{k}]\cdot[2t\hat{i} + 3t^2\hat{j} - 2\hat{k}]$

$= 2t^3+2t^7-6t^5-2t^3+4t$

$\int_0^2 (2t^3+2t^7-6t^5-2t^3+4t)\,dt = \left(\frac{t^4}{2}+\frac{1}{4}t^8-t^6-\frac{1}{2}t^4+2t^2\right)\Big|_0^2$

$= 8+64-64-8+8 = 8$

(7 points) *12.* Evaluate $\int_C (4x+3y^2)\,ds$ where C is the straight line segment from $(2,3)$ to $(1,5)$.

$x = 2-t$ $\quad 0 \le t \le 1$

$y = 3+2t$

$\int_0^1 (4(2-t)+3(3+2t)^2)\sqrt{(-1)^2+(2)^2}\,dt = \int_0^1 [8-4t+3(9+12t+4t^2)]\sqrt{5}\,dt$

$= \int_0^1 (8-4t+27+36t+12t^2)\sqrt{5}\,dt = \sqrt{5}\int_0^1 (12t^2+32t+35)\,dt$

$= \sqrt{5}\,(4t^3+16t^2+35)\Big|_0^1 = \sqrt{5}\,(4+16+35) = 55\sqrt{5}$

13. Given the transformation: $u = x + 2y$, $v = x - y$.

(3 points) *a.* Solve the system for x and y in terms of u and v.

$x = u - 2y$

$v = u - 2y - y$

$3y = u - v$

$y = \frac{1}{3}u - \frac{1}{3}v$

$x = u - 2\left(\frac{1}{3}u - \frac{1}{3}v\right)$

$x = u - \frac{2}{3}u + \frac{2}{3}v$

$x = \frac{1}{3}u + \frac{2}{3}v$

(3 points) *b.* Find the Jacobian, $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,u)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

(5 points) *c.* Find the image under the given transformation of the triangular region bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$.

$y = 0$: $\frac{1}{3}u - \frac{1}{3}v = 0 \Rightarrow u = v$

$y = x$: $\frac{1}{3}u - \frac{1}{3}v = \frac{1}{3}u + \frac{2}{3}v \Rightarrow v = 0$

$x + 2y = 2$: $\frac{1}{3}u + \frac{2}{3}v + 2\left(\frac{1}{3}u - \frac{1}{3}v\right) = 2 \Rightarrow u = 2$

14. Given $\bar{F} = (\sin y - y\sin x)\hat{i} + (\cos x + x\cos y - y)\hat{j}$.

(3 points) *a.* Verify $\bar{F}$ is a conservative vector field.

$$\frac{\partial P}{\partial y} = \cos y - \sin x$$

$$\frac{\partial Q}{\partial x} = -\sin x + \cos y$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\therefore \vec{F}$ is conservative.

(5 points) *b.* Find the potential function.

$$f(x,y) = \int P\,dx = \int (\sin y - y\sin x)\,dx = x\sin y + y\cos x + g(y)$$

$$\frac{\partial f}{\partial y} = x\cos y + \cos x + g'(y) = Q$$

$$x\cos y + \cos x + g'(y) = \cos x + x\cos y - y$$

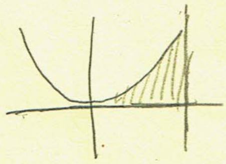
$$g'(y) = -y$$

$$g(y) = -\frac{y^2}{2} + C$$

$$f(x,y) = x\sin y + y\cos x - \frac{y^2}{2} + C$$

15. Given a thin plate of constant density $\delta = 3$ bounded by the *x*-axis, $y = x^2$, and $x = 1$.

(3 points) *a.* <u>Set up</u> the integral that would compute M_x.



$$M_x = \int_0^1 \int_0^{x^2} 3y\,dy\,dx$$

(3 points) *b.* <u>Set up</u> the integral that would compute I_y.

$$I_y = \int_0^1 \int_0^{x^2} 3x^2\,dy\,dx$$