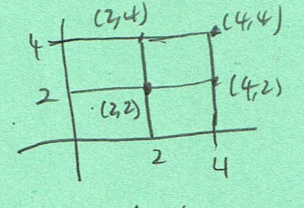
MATH 280 – EXAM #3
Winter Session 2020

Name: KEY

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes or the textbook. You may not use any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam. You may not use a graphing calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(5 points) 1. Estimate $\iint\limits_R \left(x^2+xy\right)dA$ where $R=[0,4]\times[0,4]$ by dividing R into four equal squares and evaluating the function at the upper right corner of each square.



$\Delta A = 4$

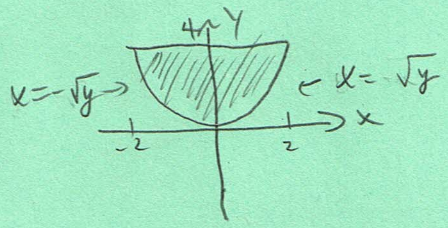
$A \approx f(2,2)\Delta A + f(2,4)\Delta A + f(4,2)\Delta A + f(4,4)\Delta A$

$= 8\cdot 4 + 12\cdot 4 + 24\cdot 4 + 32\cdot 4$

$= 304$

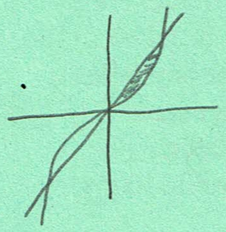
(5 points) 2. Sketch the region of integration and reverse the order of integration.

$$\int_0^2\int_{x^2}^4 f(x,y)\,dy\,dx$$



$$\int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy$$

(9 points) *3.* Evaluate the double integral $\iint_D \left(x^2+2y\right)dA$ where D is the region bounded by $y=x$, $y=x^3$, $y \geq 0$.

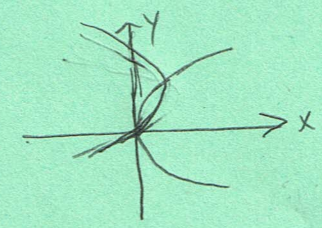


$$\int_0^1 \int_{x^3}^{x} (x^2+2y)\,dy\,dx = \int_0^1 (x^2y+y^2)\Big|_{x^3}^{x} dx$$

$$= \int_0^1 [(x^3+x^2)-(x^5+x^6)]\,dx = \int_0^1 (x^3+x^2-x^5-x^6)\,dx$$

$$= \left(\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^6}{6}-\frac{x^7}{7}\right)\Big|_0^1 = \frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{7} = \frac{23}{84}$$

(9 points) *4.* Use a double integral to find the area bounded by $x=y^2$ and $x=2y-y^2$



$$\int_0^1 \int_{y^2}^{2y-y^2} dx\,dy = \int_0^1 (2y-y^2-y^2)\,dy$$

$$= \int_0^1 (2y-2y^2)\,dy = \left(y^2-\frac{2}{3}y^3\right)\Big|_0^1$$

$$= 1-\frac{2}{3} = \frac{1}{3}$$

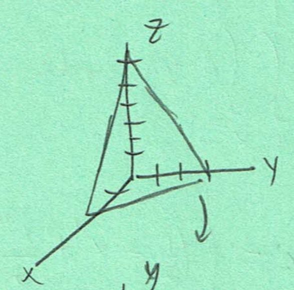
$y^2 = 2y-y^2$

$2y^2-2y=0$

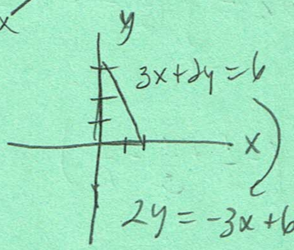
$2y(y-1)=0$

$y=0,1$

(9 points) *5.* Find the volume of the solid of the tetrahedron bounded by the coordinate planes and the plane $3x+2y+z=6$.



$$\int_0^2 \int_0^{-\frac{2}{3}x+3} \int_0^{6-3x-2y} dz\,dy\,dx = \int_0^2 \int_0^{-\frac{2}{3}x+3} z\Big|_0^{6-3x-2y} dy\,dx$$



$$= \int_0^2 \int_0^{-\frac{2}{3}x+3} (6-3x-2y)\,dy\,dx = \int_0^2 (6y-3xy-y^2)\Big|_0^{-\frac{2}{3}x+3} dx$$

$$= \int_0^2 \left[6\left(-\tfrac{2}{3}x+3\right) - 3x\left(-\tfrac{2}{3}x+3\right) - \left(-\tfrac{2}{3}x+3\right)^2\right] dx$$

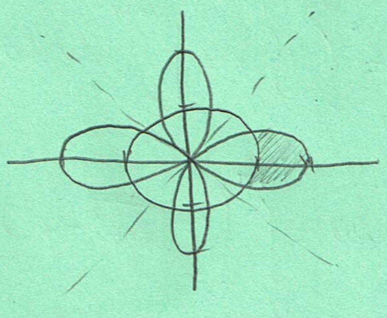
$2y = -3x+6$

$y = -\frac{2}{3}x+3$

$$= \int_0^2 \left(-4x+18+2x^2-9x-\tfrac{4}{9}x^2+4x-9\right) dx = \int_0^2 \left(\tfrac{14}{9}x^2-9x+9\right) dx$$

$$= \left(\tfrac{14}{27}x^3 - \tfrac{9}{2}x^2 + 9x\right)\Big|_0^2 = \frac{112}{27} - 18 + 18 = \frac{112}{27}$$

(9 points) *6.* Use a double integral to find the area of the region inside one leaf of the graph $r = 2\cos 2\theta$ and outside $r = 1$.



$$\int_{-\pi/6}^{\pi/6} \int_1^{2\cos 2\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6} \frac{r^2}{2}\Big|_1^{2\cos 2\theta} d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left[(2\cos 2\theta)^2 - (1)^2\right] d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(4\cos^2 2\theta - 1\right) d\theta$$

$2\cos 2\theta = 1$

$\cos 2\theta = 1/2$

$2\theta = \pi/3$

$\theta = \pi/6$

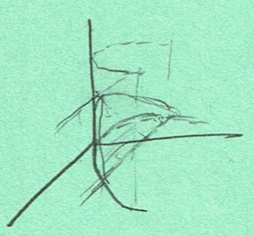
$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left[4\left(\frac{1+\cos 4\theta}{2}\right) - 1\right] d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(2 + 2\cos 4\theta - 1\right) d\theta$$

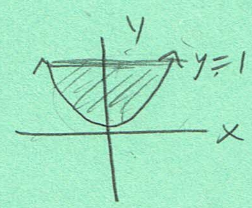
$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(1 + 2\cos 4\theta\right) d\theta = \frac{1}{2}\left(\theta + \tfrac{1}{2}\sin 4\theta\right)\Big|_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{6} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - \left(-\frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2}\right)\right] = \frac{1}{2}\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4} + \frac{\pi}{6} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right] = \frac{\pi}{6} + \frac{\sqrt{3}}{4}$$

(9 points) 7. Use a triple integral to find the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.





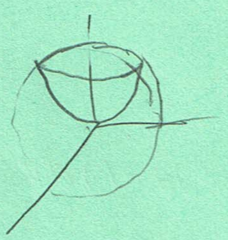
$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} z\Big|_{0}^{1-y} dy\,dx = \int_{-1}^{1}\int_{x^2}^{1}(1-y)\,dy\,dx$$

$$= \int_{-1}^{1}\left(y-\frac{y^2}{2}\right)\Big|_{x^2}^{1} dx = \int_{-1}^{1}\left[\left(1-\frac{1}{2}\right)-\left(x^2-\frac{x^4}{2}\right)\right]dx = \int_{-1}^{1}\left(\frac{1}{2}-x^2+\frac{x^4}{2}\right)dx$$

$$= \left(\frac{1}{2}x-\frac{x^3}{3}+\frac{x^5}{10}\right)\Big|_{-1}^{1} = \left(\frac{1}{2}-\frac{1}{3}+\frac{1}{10}\right)-\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{10}\right)$$

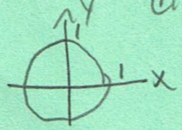
$$= \frac{8}{15}$$

(9 points) 8. Use cylindrical coordinates to find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.



paraboloid
$z = r^2$
sphere
$r^2 + z^2 = 2$
$z = \sqrt{2-r^2}$

$z = x^2+y^2 \quad x^2+y^2+z^2 = 2$
$z + z^2 = 2$
$z^2 + z - 2 = 0$
$(z+2)(z-1) = 0$
$z = -2, z = 1$
Intersects at $z = 1$
$x^2 + y^2 = 1 \leftarrow$ intersection circle



Region in xy-plane

$$\int_{0}^{2\pi}\int_{0}^{1}\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{1} rz\Big|_{r}^{\sqrt{2-r^2}} dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{1} r\left(\sqrt{2-r^2}-r\right)dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{1}\left(r\sqrt{2-r^2}-r^2\right)dr\,d\theta$$

$$= \int_{0}^{2\pi}\left(-\frac{1}{2}\cdot\frac{2}{3}(2-r^2)^{3/2}-\frac{r^3}{3}\right)\Big|_{0}^{1} d\theta = \int_{0}^{2\pi}\left[\left(-\frac{1}{3}-\frac{1}{3}\right)-\left(-\frac{2^{3/2}}{3}\right)\right]d\theta$$

$$= \int_{0}^{2\pi}\left(-\frac{2}{3}+\frac{2\sqrt{2}}{3}\right)d\theta = \left(-\frac{2}{3}+\frac{2\sqrt{2}}{3}\right)\theta\Big|_{0}^{2\pi}$$

$$= \left(-\frac{2}{3}+\frac{2\sqrt{2}}{3}\right)2\pi = \frac{4}{3}\pi\left(\sqrt{2}-1\right)$$

(9 points) *9.* Use spherical coordinates to evaluate $\iiint_D \left(x^2+y^2\right)dV$, where D lies between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

$\rho = 2 \qquad \rho = 3$

$$\int_0^{2\pi}\int_0^{\pi}\int_2^{3}\left(\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\int_2^{3}\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_2^{3}\rho^2\sin^2\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\int_2^{3}\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{\rho^5}{5}\sin^3\phi\Big|_2^3\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\left(\frac{243}{5}-\frac{32}{5}\right)\sin^3\phi\,d\phi\,d\theta = \frac{211}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\,d\phi\,d\theta = \frac{211}{5}\int_0^{2\pi}\int_0^{\pi}\sin\phi\sin^2\phi\,d\phi\,d\theta$$

$$=\frac{211}{5}\int_0^{2\pi}\int_0^{\pi}\sin\phi\left(1-\cos^2\phi\right)d\phi\,d\theta = \frac{211}{5}\int_0^{2\pi}\int_0^{\pi}\left(\sin\phi-\cos^2\phi\sin\phi\right)d\phi\,d\theta = \frac{211}{5}\int_0^{2\pi}\left(-\cos\phi+\frac{\cos^3\phi}{3}\right)\Big|_0^{\pi}d\theta$$

$$=\frac{211}{5}\int_0^{2\pi}\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]d\theta = \frac{211}{5}\int_0^{2\pi}\frac{4}{3}\,d\theta = \frac{844}{15}\int_0^{2\pi}d\theta = \frac{844}{15}\theta\Big|_0^{2\pi} = \frac{1688\pi}{15}$$

(9 points) *10.* Find the surface area of the part of the surface $z = xy$ that lies within cylinder $x^2+y^2=1$.

$f_x = y \quad f_y = x$

$$\iint_D \sqrt{y^2+x^2+1}\,dA$$

D is the circle of radius 1.

polar
$$=\int_0^{2\pi}\int_0^{1}\sqrt{r^2+1}\,r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{2}{3}\left(r^2+1\right)^{3/2}\Big|_0^1 d\theta = \frac{1}{3}\int_0^{2\pi}\left(2^{3/2}-1\right)d\theta$$

$$=\frac{2^{3/2}-1}{3}\int_0^{2\pi}d\theta = \left(\frac{2^{3/2}-1}{3}\right)\theta\Big|_0^{2\pi} = \frac{2\pi}{3}\left(2^{3/2}-1\right)$$

11. Given the following integral: $\int_0^1\int_0^{2x}(x+y)^2\sqrt{2x-y}\,dy\,dx$ and the substitutions $u=x+y$ and $v=2x-y$.

(4 points) *a.* Solve for *u* and *v* in terms of *x* and *y*.

$u=x+y$

$x=u-y$

$v=2(u-y)-y$

$v=2u-2y-y$

$v=2u-3y$

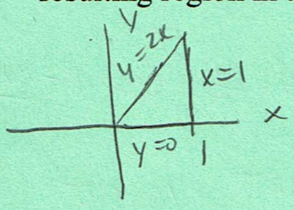
$3y=2u-v$

$y=\frac{2}{3}u-\frac{1}{3}v$

$x=u-\left(\frac{2}{3}u-\frac{1}{3}v\right)$

$x=\frac{1}{3}u+\frac{1}{3}v$

(5 points) *b.* Transform the region of integration from the *xy*-plane to the *uv*-plane. Sketch the resulting region in the *uv*-plane.



$y=0 \Rightarrow \frac{2}{3}u-\frac{1}{3}v=0 \Rightarrow \frac{2}{3}u=\frac{1}{3}v \Rightarrow v=2u$

$x=1 \Rightarrow \frac{1}{3}u+\frac{1}{3}v=1 \Rightarrow u+v=3 \Rightarrow v=-u+3$

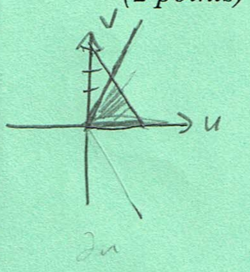
$y=2x \Rightarrow \frac{2}{3}u-\frac{1}{3}v=2\left(\frac{1}{3}u+\frac{1}{3}v\right) \Rightarrow \frac{2}{3}u-\frac{1}{3}v=\frac{2}{3}u+\frac{2}{3}v$

$\Rightarrow v=0$

(3 points) *c.* Find the Jacobian, $\frac{\partial(x,y)}{\partial(u,v)}$.

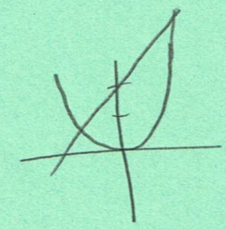
$$\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{vmatrix}=-\frac{1}{9}-\frac{2}{9}=-\frac{1}{3}$$

(2 points) *d.* Write the transformed integral in the *uv*-plane. Do not evaluate the integral.



$$\int_0^2\int_{\frac{1}{2}v}^{-v+3} u^2 v^{1/2}\,du\,dv$$

12. Given a thin plate of density $\rho(x,y) = 2x^2$ that occupies the region bounded by the $y = x+2$ and $y = x^2$.



$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1, 2$

(3 points) *a.* Set up the integral that would compute M_x.

$$\int_{-1}^{2}\int_{x^2}^{x+2} y(2x^2)\,dy\,dx$$

(3 points) *b.* Set up the integral that would compute I_y.

$$\int_{-1}^{2}\int_{x^2}^{x+2} x^2(2x^2)\,dy\,dx$$