MATH 260 – EXAM #3 KEY

(7pts)

#1) $SA = 2xy + 2xz + 2yz$ $\qquad xyz = 1000$

$z = \frac{1000}{xy}$

$SA = 2xy + \frac{2000}{y} + \frac{2000}{x}$

$A_x = 2y - 2000x^{-2}$ $\qquad A_y = 2x - 2000y^{-2}$

$2y - \frac{2000}{x^2} = 0$ $\qquad 2x - \frac{2000}{y^2} = 0$

$2y = \frac{2000}{x^2}$ $\qquad 2x = \frac{2000}{y^2}$

$x = \frac{1000}{y^2}$

$2y = \frac{2000}{\left(\frac{1000}{y^2}\right)^2}$ $\qquad x = \frac{1000}{10^2}$

$2y = \frac{2000}{(1000)(1000)} y^4$ $\qquad x = 10$

$1000y = y^4$ $\qquad z = \frac{1000}{(10)(10)} = 10$

$0 = y^4 - 1000y$

$0 = y(y^3 - 1000)$ $\qquad x = 10\text{ cm},\ y = 10\text{ cm},\ z = 10\text{ cm}$

$y = 0 \quad y^3 = 1000$

$y = 10$

$A_{xx} = 4000x^{-3}$ $\quad A_{xy} = 2$ $\quad A_{yy} = 4000y^{-3}$

$A_{xx}A_{yy} - (A_{xy})^2 = (4)(4) - 2^2 = 16 - 4 = 12 > 0$ $\quad 4 > 0$ local min

(7pts)

#2) $f(x,y,z) = x + 2y + 3z$ $\qquad g(x,y,z) = x^2 + y^2 + z^2 = 25$

$\vec{\nabla} f = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ $\qquad \vec{\nabla} g = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$

$\vec{\nabla} f = \lambda \vec{\nabla} g$ $\qquad 1 = \lambda 2x$ $\quad 2 = \lambda 2y$ $\quad 3 = \lambda 2z$ $\quad x^2 + y^2 + z^2 = 25$

$x = \frac{1}{2\lambda}$ $\quad y = \frac{2}{2\lambda}$ $\quad z = \frac{3}{2\lambda}$ $\quad \left(\frac{1}{4\lambda^2}\right) + \left(\frac{4}{4\lambda^2}\right) + \left(\frac{9}{4\lambda^2}\right) = 25$

$\frac{14}{4\lambda^2} = 25$

$\frac{14}{100} = \lambda^2$

$\lambda = \pm \frac{\sqrt{14}}{10}$

$\lambda = \frac{\sqrt{14}}{10}$ $\quad x = \frac{5}{\sqrt{14}}$ $\quad y = \frac{10}{\sqrt{14}}$ $\quad z = \frac{15}{\sqrt{14}}$

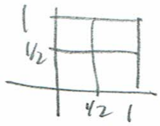
$\lambda = -\frac{\sqrt{14}}{10}$ $\quad x = -\frac{5}{\sqrt{14}}$ $\quad y = -\frac{10}{\sqrt{14}}$ $\quad z = -\frac{15}{\sqrt{14}}$

$f\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right) = \frac{5}{\sqrt{14}} + \frac{20}{\sqrt{14}} + \frac{45}{\sqrt{14}} = \frac{70}{\sqrt{14}}$

$f\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right) = -\frac{5}{\sqrt{14}} - \frac{20}{\sqrt{14}} - \frac{45}{\sqrt{14}} = -\frac{70}{\sqrt{14}}$

(5pts)

#3.)



$\Delta A = 1/4$ $\quad \Delta A\left[f(1/2,1/2)+f(\tfrac{1}{2},1)+f(1,\tfrac{1}{2})+f(1,1)\right]$

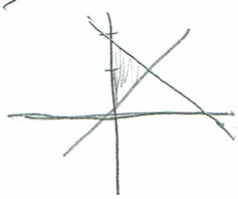
$$A \approx \frac{1}{4}\left[\left(3\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}+4\cdot\tfrac{1}{4}\cdot\tfrac{1}{2}\right)+\left(3\cdot\tfrac{1}{2}\cdot 1+4\cdot\tfrac{1}{4}\cdot 1\right)+\left(3\cdot 1\cdot\tfrac{1}{4}+4\cdot 1\cdot\tfrac{1}{2}\right)+(3+4)\right]$$

$$A \approx \frac{1}{4}\left[\frac{3}{8}+\frac{1}{2}+\frac{3}{2}+1+\frac{3}{4}+2+7\right] = \frac{1}{4}\left[12+\frac{3}{8}+\frac{3}{4}\right]$$

$$= \frac{1}{4}\left[\frac{96+3+6}{8}\right] = \frac{105}{32}$$

(7pts)

#4.)



$$\int_0^1\int_x^{2-x}(x^2+y^2)\,dy\,dx = \int_0^1 \left(\frac{y^3}{3}+x^2y\right)\Big|_x^{2-x} dx$$

$$= \int_0^1\left[\left[\frac{(2-x)^3}{3}+x^2(2-x)\right]-\left(\frac{x^3}{3}+x^3\right)\right]dx = \int_0^1\left(\frac{(2-x)^3}{3}+2x^2-\frac{7}{3}x^3\right)dx$$

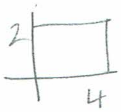
$$= \left(-\frac{(2-x)^4}{12}+\frac{2}{3}x^3-\frac{7}{12}x^4\right)\Big|_0^1 = \left(-\frac{1}{12}+\frac{2}{3}-\frac{7}{12}\right)-\left(-\frac{4}{3}\right) = 2-\frac{8}{12}$$

$$= 2-\frac{2}{3} = \frac{4}{3}$$

(7pts)

#5.)

$$\int_0^4\int_0^2 (x^2y)\,dy\,dx = \int_0^4 \frac{x^2y^2}{2}\Big|_0^2 dx = \int_0^4 2x^2\,dx = \frac{2}{3}x^3\Big|_0^4 = \frac{128}{3}$$



Area = 8 $\quad$ Average value $\dfrac{\frac{128}{3}}{8} = \dfrac{128}{3}\cdot\dfrac{1}{8} = \dfrac{16}{3}$

(7pts)

#6)

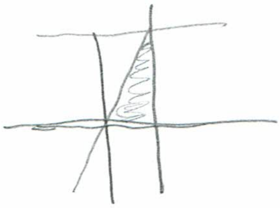
$$\iint(4-y-2x)\,dA = \int_0^{2\pi}\int_0^1(4-r\sin\theta-2r\cos\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1(4r-r^2\sin\theta-2r^2\cos\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(2r^2-\frac{r^3}{3}\sin\theta-\frac{2r^3}{3}\cos\theta\right)\Big|_0^1 d\theta = \int_0^{2\pi}\left(2-\frac{1}{3}\sin\theta-\frac{2}{3}\cos\theta\right)d\theta = \left(2\theta+\frac{1}{3}\cos\theta-\frac{2}{3}\sin\theta\right)\Big|_0^{2\pi}$$

$$= \left(4\pi+\frac{1}{3}\right)-\left(\frac{1}{3}\right) = 4\pi$$

(7pts)
#7.) $\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,dx\,dy$ $\quad x=\frac{y}{2}$

$y=2x$



$$\int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx = \int_0^1 \frac{y^2}{2}\cos(x^3-1)\Big|_0^{2x} dx = \int_0^1 2x^2\cos(x^3-1)\,dx$$

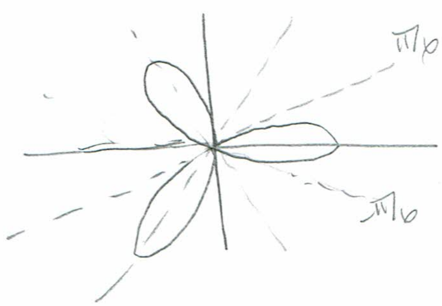
$u=x^3-1$
$du=3x^2\,dx$
$\frac{1}{3}du=x^2\,dx$

$$\frac{2}{3}\int_0^1 \cos u\,du = \frac{2}{3}\sin u\Big|_0^1 = \frac{2}{3}\sin(x^3-1)\Big|_0^1$$

$$=\frac{2}{3}\sin 0 - \frac{2}{3}\sin(-1) = \frac{2}{3}\sin 1$$

(7pts)
#8.) $r=4\cos 3\theta$

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
r	4	0	-4	0	4	0	4	0	4	0	-4	0	4

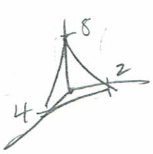


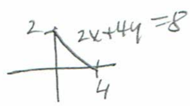
$$\int_{-\pi/6}^{\pi/6} \int_0^{4\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6} \frac{r^2}{2}\Big|_0^{4\cos 3\theta} d\theta$$

$$=\frac{1}{2}\int_{-\pi/6}^{\pi/6} 16\cos^2 3\theta\,d\theta = 8\int_{-\pi/6}^{\pi/6}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$=4\left(\theta+\frac{1}{6}\sin 6\theta\right)\Big|_{-\pi/6}^{\pi/6} = 4\left(\frac{\pi}{6}+\frac{\pi}{6}\right) = \frac{4\pi}{3}$$

(7pts)
#9.





$$\int_0^4 \int_0^{2-\frac{1}{2}x} \int_0^{8-2x-4y} dz\,dy\,dx = \int_0^4 \int_0^{2-\frac{1}{2}x} z\Big|_0^{8-2x-4y} dy\,dx$$

$4y=8-2x$
$y=2-\frac{1}{2}x$

$$=\int_0^4 \int_0^{2-\frac{1}{2}x} (8-2x-4y)\,dy\,dx = \int_0^4 (8y-2xy-2y^2)\Big|_0^{2-\frac{1}{2}x} dx$$

$$=\int_0^4 \left[8(2-\tfrac{1}{2}x)-2x(2-\tfrac{1}{2}x)-2(2-\tfrac{1}{2}x)^2\right]dx = \int_0^4 (16-4x-4x+x^2+8-4x-\tfrac{1}{2}x^2)\,dx = \int_0^4 (\tfrac{1}{2}x^2-12x+24)\,dx$$

$$=\left(\frac{1}{6}x^3-6x^2+24x\right)\Big|_0^4 = \frac{64}{6}-96+96 = \frac{32}{3}$$

(7pts)
#10)

$$\int_0^\pi\int_0^2\int_0^{r\sin\theta} r\sin\theta\, z\, r\,dz\,dr\,d\theta = \int_0^\pi\int_0^2\int_0^{r\sin\theta} r^2\sin\theta\, z\,dz\,dr\,d\theta$$

$$= \int_0^\pi\int_0^2 r^2\sin\theta\, \frac{z^2}{2}\Big|_0^{r\sin\theta} dr\,d\theta = \frac{1}{2}\int_0^\pi\int_0^2 r^4\sin^3\theta\,d\theta = \frac{1}{2}\int_0^\pi \frac{r^5}{5}\Big|_0^2 \sin^3\theta\,d\theta$$

$$= \frac{16}{5}\int_0^\pi \sin^3\theta\,d\theta = \frac{16}{5}\int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta = -\frac{16}{5}\int_0^\pi (1-u^2)\,du$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$$= -\frac{16}{5}\left(u-\frac{u^3}{3}\right)\Big|_0^\pi = -\frac{16}{5}\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\Big|_0^\pi$$

$$= -\frac{16}{5}\left[\left(-1+\frac{1}{3}\right)-\left(1-\frac{1}{3}\right)\right] = -\frac{16}{5}\left[-\frac{2}{3}-\frac{2}{3}\right] = \left(-\frac{16}{5}\right)\left(-\frac{4}{3}\right) = \frac{64}{15}$$

(7pts)
#11)

$$\int_0^\pi\int_0^\pi\int_0^3 \rho^2\sin^2\phi\,\sin^2\theta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^\pi\int_0^\pi\int_0^3 \rho^4\sin^3\phi\,\sin^2\theta\,d\rho\,d\phi\,d\theta = \int_0^\pi\int_0^\pi \frac{\rho^5}{5}\sin^3\phi\,\sin^2\theta\Big|_0^3 d\phi\,d\theta = \frac{243}{5}\int_0^\pi\int_0^\pi \sin^3\phi\,\sin^2\theta\,d\phi\,d\theta$$

$$= \frac{243}{5}\int_0^\pi\int_0^\pi (1-\cos^2\phi)\sin\phi\,\sin^2\theta\,d\phi\,d\theta = -\frac{243}{5}\int_0^\pi\int_0^\pi (1-u^2)\sin^2\theta\,du\,d\theta$$

$u = \cos\phi$

$du = -\sin\phi\,d\theta$

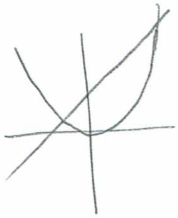
$-du = \sin\phi\,d\theta$

$$= -\frac{243}{5}\int_0^\pi \left(u-\frac{u^3}{3}\right)\Big|_0^\pi \sin^2\theta\,d\theta = -\frac{243}{5}\int_0^\pi \left(\cos\phi-\frac{\cos^3\phi}{3}\right)\Big|_0^\pi \sin^2\theta\,d\theta$$

$$= -\frac{243}{5}\int_0^\pi \left(\left(-1+\frac{1}{3}\right)-\left(1-\frac{1}{3}\right)\right)\sin^2\theta\,d\theta = -\frac{243}{5}\left(-\frac{4}{3}\right)\int_0^\pi \sin^2\theta\,d\theta = \frac{324}{5}\int_0^\pi \left(\frac{1-\sin 2\theta}{2}\right)d\theta$$

$$= \frac{324}{5}\left(\frac{1}{2}\theta+\frac{1}{4}\cos 2\theta\right)\Big|_0^\pi = \frac{324}{5}\left(\left(\frac{\pi}{2}+\frac{1}{4}\right)-\left(0+\frac{1}{4}\right)\right) = \frac{162\pi}{5}$$

#12)



$y = x^2 \quad y = x+2$

$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

2pts

a.) $M_x = \int_{-1}^{2} \int_{x^2}^{x+2} x^2 y \, dy \, dx$

2pts

b.) $M_y = \int_{-1}^{2} \int_{x^2}^{x+2} x^3 \, dy \, dx$

2pts

c.) $M = \int_{-1}^{2} \int_{x^2}^{x+2} x^2 \, dy \, dx$

2pts

d) $I_x = \int_{-1}^{2} \int_{x^2}^{x+2} x^2 y^2 \, dy \, dx$

2pts

e.) $I_y = \int_{-1}^{2} \int_{x^2}^{x+2} x^4 \, dy \, dx$