MATH 280 – EXAM #1 **Name:** KEY
Fall Semester 2018

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. You will need your Mt. SAC student identification card to submit this exam. Good luck!

1. Given the following points: $P(4.-2,5)$ and $Q(3,1,4)$.

(3 points) *a.* Determine the distance between the points.

$$d=\sqrt{(3-4)^2+(1+2)^2+(4-5)^2}$$

$$=\sqrt{(-1)^2+(3)^2+(-1)^2}$$

$$=\sqrt{1+9+1}=\sqrt{11}$$

(3 points) *b.* Find the equation of the line segment between P and Q.

$$\vec{PQ}=\langle -1,3,-1\rangle$$

$$x=4-t$$

$$y=-2+3t \qquad 0\le t\le 1$$

$$z=5-t$$

(5 points) *2.* Find the parametric form for the equation of tangent line at the point $(2,\ln 4,1)$ to the curve traced by the vector function $\vec{r}(t)=\left(\sqrt{t^2+3}\right)\hat{i}+\left[\ln\left(t^2+3\right)\right]\hat{j}+(t)\hat{k}$.

$\uparrow$
$t=1$

$$\vec{r}\,'(t)=\frac{2t}{2(t^2+3)^{1/2}}\hat{i}+\frac{2t}{t^2+3}\hat{j}+\hat{k}$$

$$\vec{r}\,'(1)=\frac{1}{(4)^{1/2}}\hat{i}+\frac{2}{4}\hat{j}+\hat{k}=\frac{1}{2}\hat{i}+\frac{1}{2}\hat{j}+\hat{k}$$

$$x=2+\frac{1}{2}t$$

$$y=\ln 4+\frac{1}{2}t$$

$$z=1+t$$

3. Given the following vectors: $\vec{a} = \langle 1,3,-4\rangle$, $\vec{b} = \langle 2,1,-3\rangle$, and $\vec{c} = \langle 4.2,-5\rangle$. Determine the following:

(2 points) a. $4\vec{b} - 3\vec{c}$

$4\langle 2,1,-3\rangle - 3\langle 4,2,-5\rangle$

$= \langle 8,4,-12\rangle + \langle -12,-6,15\rangle$

$= \langle -4,-2,3\rangle$

(2 points) b. $|\vec{a}|$

$\sqrt{1^2+3^2+(-4)^2}$

$= \sqrt{1+9+16} = \sqrt{26}$

(3 points) c. $\vec{a}\cdot\vec{c}$

$\langle 1,3,-4\rangle \cdot \langle 4,2,-5\rangle$

$= 4+6+20$

$= 30$

(2 points) d. The unit vector in the direction of $\vec{b}$

$\frac{\vec{b}}{|\vec{b}|} = \frac{\langle 2,1,-3\rangle}{\sqrt{2^2+1^2+(-3)^2}} = \frac{\langle 2,1,-3\rangle}{\sqrt{4+1+9}}$

$= \frac{\langle 2,1,-3\rangle}{\sqrt{14}} = \langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\rangle$

(4 points) e. $\vec{a}\times\vec{b}$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -4 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i}(-9+4) - \hat{j}(-3+8) + \hat{k}(1-6)$

$= -5\hat{i} - 5\hat{j} - 5\hat{k}$

(3 points) f. $\text{proj}_{\vec{a}}\vec{c}$

$\left(\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|^2}\right)\vec{a} = \frac{30}{26}\langle 1,3,-4\rangle$

$= \frac{15}{13}\langle 1,3,-4\rangle$

$= \langle \frac{15}{13}, \frac{45}{13}, -\frac{60}{13}\rangle$

(2 points) g. The area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

$|\vec{a}\times\vec{b}| = \sqrt{(-5)^2+(-5)^2+(-5)^2} = \sqrt{25+25+25} = \sqrt{75}$

$= 5\sqrt{3}$ square units.

(5 points) 4. Find the parametric form for the equation of the line of intersection of the planes $x+2y-9z=7$ and $2x-3y+17z=0$

$\vec{n}_1 = \langle 1, 2, -9 \rangle \quad \vec{n}_2 = \langle 2, -3, 17 \rangle$

$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -9 \\ 2 & -3 & 17 \end{vmatrix} = \hat{i}(34-27) - \hat{j}(17+18) + \hat{k}(-3-4)$

$= 7\hat{i} - 35\hat{j} - 7\hat{k}$

$x+2y-9z=7$

$2x-3y+17z=0$

let $z=0$ $\quad -2(x+2y=7) \Rightarrow -2x-4y=-14$

$2x-3y=0 \qquad 2x-3y=0$

$-7y=-14$

$y=2$

$x+2(2)=7$

$x=3$

$(3,2,0)$

$x=3+7t$

$y=2-35t$

$z=-7t$

(5 points) 5. Find the equation of the plane that passes through $P(3,4,-1)$, $Q(2,4,-3)$, and $R(-1,4,2)$.

$\vec{PQ} = \langle -1, 0, -2 \rangle$

$\vec{PR} = \langle -4, 0, 3 \rangle$

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -2 \\ -4 & 0 & 3 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(-3-8) + \hat{k}(0-0) = +11\hat{j}$

$0(x-3) + 11(y-4) + 0(z+1) = 0$

$11y - 44 = 0$

$y = 4$

(4 points) 6. A sled is pulled along a level path by a rope. A 50-lb force acting at an angle of 60° above the horizontal moves the sled 40 feet. Find the work done by the force.

$\vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos\theta$

$= (50\,lb)(40\,ft)\cos 60°$

$= 2000\left(\frac{1}{2}\right) = 1000$ ft-lb.

(4 points) 7. Find the volume of the parallelepiped determined by the vectors $\vec{a}=\langle 3,1,2\rangle$, $\vec{b}=\langle 1,-2,1\rangle$, and $\vec{c}=\langle -4,1,1\rangle$.

$$\begin{vmatrix} 3 & 1 & 2 \\ 1 & -2 & 1 \\ -4 & 1 & 1 \end{vmatrix} = 3(-2-1)-1(1+4)+2(1-8)$$
$$= 3(-3)-1(5)+2(-7)$$
$$= -9-5-14$$
$$= -28$$

Volume = 28 cubic units

(3 points) 8. Evaluate the following limit: $\lim_{t\to\pi/6} \vec{r}(t)$ where $\vec{r}(t)=(\sec t)\hat{i}+t^2\hat{j}+(-\sin 2t)\hat{k}$

$$\lim_{t\to\pi/6} \vec{r}(t) = \left(\sec\frac{\pi}{6}\right)\hat{i} + \left(\frac{\pi}{6}\right)^2\hat{j} + \left(-\sin\frac{2\pi}{6}\right)\hat{k}$$
$$= \frac{2}{\sqrt{3}}\hat{i} + \frac{\pi^2}{36}\hat{j} - \frac{\sqrt{3}}{2}\hat{k}$$

(5 points) 9. Find the length of the curve: $\vec{r}(t)=(3\cos t)\hat{i}+(3\sin t)\hat{j}+2t^{3/2}\hat{k}$; $0\le t\le 3$

$$\int_0^3 \sqrt{(-3\sin t)^2+(3\cos t)^2+(3t^{1/2})^2}\,dt$$
$$= \int_0^3 \sqrt{9\sin^2 t + 9\cos^2 t + 9t}\,dt = \int_0^3 \sqrt{9+9t}\,dt$$
$$= \int_0^3 3\sqrt{1+t}\,dt \qquad \begin{array}{l} u=1+t \\ du=dt \end{array} \qquad 3\int_0^3 u^{1/2}\,du = 3\cdot\frac{2}{3}u^{3/2}\Big|_0^3$$
$$= 2(1+t)^{3/2}\Big|_0^3 = 2\left[(1+3)^{3/2}-(1+0)^{3/2}\right]$$
$$= 2\left[4^{3/2}-1^{3/2}\right] = 2[8-1] = 14$$

10. Given the position vector: $\vec{r}(t) = (6\sin 2t)\hat{i} + (6\cos 2t)\hat{j} + 5t\hat{k}$.

(4 points) *a.* Find the unit tangent vector $\hat{T}(t)$.

$\vec{r}\,'(t) = (12\cos 2t)\hat{i} + (-12\sin 2t)\hat{j} + 5\hat{k}$

$|\vec{r}\,'(t)| = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25}$

$= \sqrt{144(\cos^2 2t + \sin^2 2t) + 25}$

$= \sqrt{144+25} = \sqrt{169} = 13$

$\hat{T}(t) = \frac{12}{13}\cos 2t\,\hat{i} - \frac{12}{13}\sin 2t\,\hat{j} + \frac{5}{13}\hat{k}$

(4 points) *b.* Find the unit normal vector $\hat{N}(t)$.

$\hat{T}'(t) = -\frac{24}{13}\sin 2t - \frac{24}{13}\cos 2t$

$|\hat{T}'(t)| = \sqrt{\left(-\frac{24}{13}\sin 2t\right)^2 + \left(-\frac{24}{13}\cos 2t\right)^2}$

$= \sqrt{\left(\frac{24}{13}\right)^2 \sin^2 2t + \left(\frac{24}{13}\right)^2 \cos^2 2t} = \frac{24}{13}$

$\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|} = -\sin 2t\,\hat{i} - \cos 2t\,\hat{j} + 0\hat{k}$

(4 points) *c.* Find the binormal vector $\hat{B}(t)$.

$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{12}{13}\cos 2t & -\frac{12}{13}\sin 2t & \frac{5}{13} \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix}$

$= \hat{i}\left(0 + \frac{5}{13}\cos 2t\right) - \hat{j}\left(0 + \frac{5}{13}\sin 2t\right) + \hat{k}\left(-\frac{12}{13}\cos^2 2t - \frac{12}{13}\sin^2 2t\right)$

$= \left(\frac{5}{13}\cos 2t\right)\hat{i} + \left(-\frac{5}{13}\sin 2t\right)\hat{j} + \left(-\frac{12}{13}\right)\hat{k}$

(4 points) *d.* Find the curvature κ.

$\kappa = \frac{|\hat{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{24/13}{13} = \frac{24}{169}$

11. Evaluate the following integrals.

$\int t\cos 2t\, dt$

$u = t \qquad dv = \cos 2t\, dt$

$du = dt \qquad v = \frac{1}{2}\sin 2t$

$\int t\cos 2t\, dt = \frac{1}{2}t\sin 2t - \int \frac{1}{2}\sin 2t\, dt$

$= \frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t + C$

(4 points) a. $\int_0^1 \left[\left(t^3\right)\hat{i} + \left(t\cos 2t\right)\hat{j} + \left(\sin 2t\right)\hat{k}\right]dt$

$\int_0^1 [(t^3)\hat{i} + (t\cos 2t)\hat{j} + (\sin 2t)\hat{k}]\, dt$

$= \frac{t^4}{4}\hat{i} + \left[\left(\frac{1}{2}t\sin 2t\right) + \frac{1}{4}\cos 2t\right]\hat{j} + \left(-\frac{1}{2}\cos 2t\right)\hat{k} \Big|_0^1$

$= \frac{1}{4}\hat{i} + \left[\frac{1}{2}\sin 2 + \frac{1}{4}\cos 2 - \frac{1}{4}\right]\hat{j} + \left(-\frac{1}{2}\cos 2 + \frac{1}{2}\right)\hat{k}$

(4 points) b. $\int \left[\left(\frac{1}{t}\right)\hat{i} + \left(t^5\right)\hat{j} + \left(e^{3t}\right)\hat{k}\right]dt$

$= \ln|t|\,\hat{i} + \frac{t^6}{6}\hat{j} + \frac{1}{3}e^{3t}\hat{k} + \vec{C}$

(5 points) 12. Given the vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$, and c and d as scalars. Prove the following vector property: $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

$(c+d)\vec{u} = (c+d)\langle u_1, u_2, u_3\rangle = \langle (c+d)u_1, (c+d)u_2, (c+d)u_3\rangle$

$= \langle cu_1 + du_1, cu_2 + du_2, cu_3 + du_3\rangle$

$= \langle cu_1, cu_2, cu_3\rangle + \langle du_1, du_2, du_3\rangle$

$= c\langle u_1, u_2, u_3\rangle + d\langle u_1, u_2, u_3\rangle$

$= c\vec{u} + d\vec{u}$

(5 points) *13.* Given the vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Prove the following vector property: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$

$= u_1 v_1 + u_2 v_2 + u_3 v_3$

$= v_1 u_1 + v_2 u_2 + v_3 u_3$

$= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle$

$= \vec{v} \cdot \vec{u}$

(5 points) *14.* Prove the following: If $|\vec{r}(t)| = c$, then $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal for all t.

$|\vec{r}(t)| = \vec{c} \Rightarrow |\vec{r}(t)|^2 = c^2 \Rightarrow \vec{r}(t) \cdot \vec{r}(t) = c^2$

$\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{d}{dt}(c^2)$

$\vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0$

$2\vec{r}(t) \cdot \vec{r}'(t) = 0$

$\vec{r}(t) \cdot \vec{r}'(t) = 0$

$\therefore$ $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.

15. Graph the following surfaces on the given sheet of triangular graph paper.

(4 points) *a.* $\frac{z^2}{36} = \frac{x^2}{16} + \frac{y^2}{4}$

(4 points) *b.* $\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{16} = 1$