MATH 280 – EXAM #2 **Name:** KEY
Fall Semester 2018

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There are 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes, the textbook, or any unauthorized sources for assistance during this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You may not use a calculator on this exam. Good luck!

1. Given the following function: $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

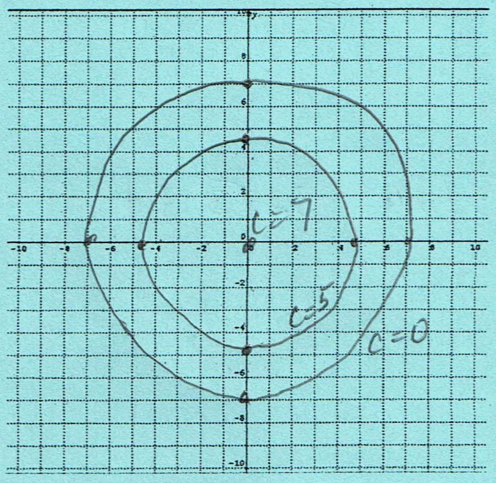
(3 points) *a.* Find the function's domain.

$x \neq 0$

(3 points) *b.* Find the function's range.

$(-\pi/2, \pi/2)$

(3 points) 2. Given the following function: $f(x,y) = \sqrt{49 - x^2 - y^2}$. Sketch the function's level curves when $c = 0,\ c = 5,\ c = 7$.



$c = 0$
$\sqrt{49 - x^2 - y^2} = 0$
$49 - x^2 - y^2 = 0$
$x^2 + y^2 = 49$

$c = 5$
$\sqrt{49 - x^2 - y^2} = 5$
$49 - x^2 - y^2 = 25$
$24 = x^2 + y^2$

$c = 7$
$\sqrt{49 - x^2 - y^2} = 7$
$49 - x^2 - y^2 = 49$
$x^2 + y^2 = 0$

3. A baseball is thrown at a speed of 32 ft/s from the stands 32 ft above the field at an angle of $30°$ up from the horizontal.

(3 points) *a.* What is the position vector that models this situation? $\left(g = 32 \text{ ft/s}^2\right)$

$\vec{r}(t) = (32)\cos 30° t\,\hat{i} + \left(-\frac{1}{2}(32)t^2 + (32)\sin 30° t + 32\right)\hat{j}$

$= 32\frac{\sqrt{3}}{2} t\,\hat{i} + \left(-16t^2 + 16t + 32\right)\hat{j}$

$= 16\sqrt{3}\, t\,\hat{i} + \left(-16t^2 + 16t + 32\right)\hat{j}$

(3 points) *b.* What is the maximum height of the baseball?

$\vec{r}_y = -16t^2 + 16t + 32$

$r_y' = -32t + 16$

$-32t + 16 = 0$

$16 = 32t$

$t = 1/2$ sec

$-16\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right) + 32$

$-4 + 8 + 32$

36 ft

(3 points) *c.* When will the baseball land?

$-16t^2 + 16t + 32 = 0$

$-16\left(t^2 - \cancel{t} - 2\right) = 0$

$-16(t-2)(t+1) = 0$

$t = 2, -1$ $\quad 2 = 2$ seconds

(3 points) *d.* What is the range?

$16\sqrt{3}(2) = 32\sqrt{3}$ ft

(4 points) *4.* Show that $f(x,y)=\dfrac{x^4}{x^4-y^2}$ has no limit as $(x,y)\to(0,0)$. Use $y=kx^2$, $k\neq\pm1$ and explain the result.

$$\lim_{(x,y)\to(0,0)}\frac{x^4}{x^4-y^2}=\lim_{(x,y)\to(0,0)}\frac{x^4}{x^4-(kx^2)^2}=\lim_{(x,y)\to(0,0)}\frac{x^4}{x^4-k^2x^4}=\frac{1}{1-k^2}$$

The limit is dependent on path.

If $k=2$ the limit $=\dfrac{1}{1-2^2}=-\dfrac{1}{3}$

If $k=3$ the limit $=\dfrac{1}{1-3^2}=-\dfrac{1}{8}$

5. Find the following limits.

(3 points) *a.* $\displaystyle\lim_{\substack{(x,y)\to(2,0)\\2x-y\neq4}}\frac{\sqrt{2x-y}-2}{2x-y-4}\cdot\frac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2}=\lim_{\substack{(x,y)\to(2,0)\\2x-y\neq4}}\frac{2x-y-4}{(2x-y-4)(\sqrt{2x-y}+2)}$

$$=\lim_{\substack{(x,y)\to(2,0)\\2x-y\neq4}}\frac{1}{\sqrt{2x-y}+2}=\frac{1}{\sqrt{2(2)-0}+2}=\frac{1}{4}$$

(3 points) *b.* $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\sin\left(x^2+y^2\right)}{x^2+y^2}=1$

(3 points) c. $\lim_{(x,y)\to(2,2)} \dfrac{x-y}{x^4-y^4} = \lim_{(x,y)\to(2,2)} \dfrac{x-y}{(x^2+y^2)(x^2-y^2)}$

$= \lim_{(x,y)\to(2,2)} \dfrac{\cancel{x-y}}{(x^2+y^2)(x+y)\cancel{(x-y)}}$

$= \lim_{(x,y)\to(2,2)} \dfrac{1}{(x^2+y^2)(x+y)}$

$= \dfrac{1}{(2^2+2^2)(2+2)} = \dfrac{1}{8\cdot 4} = \dfrac{1}{32}$

(6 points) 6. **Find the equation of the tangent plane and the parametric form of the normal line to the graph of the given equation at the point P_0.**

$$x^3+3x^2y^2+y^3+4xy-z^2=0; \quad P_0(1,1,3)$$

$\vec{\nabla} f = (3x^2+6xy^2+4y)\hat{i} + (6x^2y+3y^2+4x)\hat{j} + (-2z)\hat{k}$

$\vec{\nabla} f\,\big|_{P_0} = (3(1)^2+6(1)(1)^2+4(1))\hat{i}\,(6(1)^2(1)+3(1)^2+4(1))\hat{j} + (-2(3))\hat{k}$

$= 13\hat{i} + 13\hat{j} - 6\hat{k}$

$13(x-1)+13(y-1)-6(z-3)=0$

$13x-13+13y-13-6z+18=0$

$13x+13y-6z-8=0$

$x = 1+13t$

$y = 1+13t$

$z = 3-6t$

7. Given $f(x,y)=\sin(x^4y^5)$. Determine the following.

(3 points) a. f_x $= 4x^3y^5\cos(x^4y^5)$

(3 points) b. f_y $= 5x^4y^4\cos(x^4y^5)$

(3 points) c. f_{xx} $= 12x^2y^5\cos(x^4y^5) - 4x^3y^5\sin(x^4y^5)\,4x^3y^5$

$= 12x^2y^5\cos(x^4y^5) - 16x^6y^{10}\sin(x^4y^5)$

(3 points) d. f_{yx} $= 20x^3y^4\cos(x^4y^5) - 5x^4y^4\sin(x^4y^5)\,4x^3y^5$

$= 20x^3y^4\cos(x^4y^5) - 20x^7y^9\sin(x^4y^5)$

(3 points) e. f_{yy} $= 20x^4y^3\cos(x^4y^5) - 5x^4y^4\sin(x^4y^5)\,5x^4y^4$

$= 20x^4y^3\cos(x^4y^5) - 25x^8y^8\sin(x^4y^5)$

(5 points) *8.* Given $w = \ln\left(x^2+y^2+z^2\right)$, $x=\cos t$, $y=\sin t$, $z=4\sqrt{t}$. Write the chain rule that is used to find $\frac{dw}{dt}$ and then use the chain rule to find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = \frac{2x}{x^2+y^2+z^2}\cdot(-\sin t) + \frac{2y}{x^2+y^2+z^2}(\cos t) + \frac{2z}{x^2+y^2+z^2}\left(4\cdot\tfrac{1}{2}t^{-1/2}\right)$$

$$= \frac{-2\sin t\cos t + 2\cos t\sin t + 2(4t^{1/2})2t^{-1/2}}{\cos^2 t + \sin^2 t + 16t}$$

$$= \frac{16}{1+16t}$$

(2 points) *9.* Write the Chain Rule formulas that will find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for the following functions:

$$w = f(x,y,z),\ x=g(u,v),\ y=h(u,v),\ z=k(u,v)$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

10. Given the following: $f(x,y,z) = \frac{x}{y} - yz;\ P_0(4,1,1)$

(2 points) *a.* State the directions in which the function increases most rapidly at P_0.

$$\vec{\nabla} f = \left(\frac{1}{y}\right)\hat{\imath} + \left(-\frac{x}{y^2} - z\right)\hat{\jmath} + (-y)\hat{k}$$

$$\vec{\nabla} f\big|_{P_0} = \hat{\imath} - 5\hat{\jmath} - \hat{k} \qquad |\vec{\nabla} f|_{P_0}| = \sqrt{1+25+1} = 3\sqrt{3}$$

$$\text{Direction} = \frac{1}{3\sqrt{3}}\hat{\imath} - \frac{5}{3\sqrt{3}}\hat{\jmath} - \frac{1}{3\sqrt{3}}\hat{k}$$

(2 points) *a.* State the directions in which the function decreases most rapidly at P_0.

$$\text{Direction} = -\frac{1}{3\sqrt{3}}\hat{\imath} + \frac{5}{3\sqrt{3}}\hat{\jmath} + \frac{1}{3\sqrt{3}}\hat{k}$$

or $\quad -\hat{\imath} + 5\hat{\jmath} + \hat{k}$

(6 points) *11.* Use partial derivatives to find $\dfrac{dy}{dx}$ if $e^{x^3y^2} - y^3\cos x = x^4y^3$, where y is a differentiable function of x.

$F(x,y) = e^{x^3y^2} - y^3\cos x - x^4y^3 = 0$

$F_x = 3x^2y^2e^{x^3y^2} + y^3\sin x - 4x^3y^3$

$F_y = 2xy^3e^{x^3y^2} - 3y^2\cos x - 3x^4y^2$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2y^2e^{x^3y^2} + y^3\sin x - 4x^3y^3}{2x^3ye^{x^3y^2} - 3y^2\cos x - 3x^4y^2}$$

(6 points) *12.* Find the standard linearization $L(x,y)$ of the following function at the given point.

$$f(x,y) = \cos x \sin y \text{ at } (\pi/4, \pi/4)$$

$f_x = -\sin x \sin y \qquad f_y = \cos x \cos y$

$f(\pi/4,\pi/4) = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = \frac{1}{2} \quad f_x(\pi/4,\pi/4) = -(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = -\frac{1}{2} \quad f_y(\pi/4,\pi/4) = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = \frac{1}{2}$

$L(x,y) = f(\pi/4,\pi/4) + f_x(\pi/4,\pi/4)(x-\pi/4) + f_y(\pi/4,\pi/4)(y-\pi/4)$

$= \frac{1}{2} + (-\frac{1}{2})(x-\pi/4) + \frac{1}{2}(y-\pi/4)$

$= \frac{1}{2} - \frac{1}{2}x + \frac{\pi}{8} + \frac{1}{2}y - \frac{\pi}{8} = \frac{1}{2} - \frac{1}{2}x + \frac{1}{2}y$

(6 points) *13.* Find the directional derivative of the following function at P_0 in the direction of $\vec{u}$:

$$f(x,y,z) = xy + yz + zx; \quad P_0(2,3,-1); \quad \vec{u} = 3\hat{i} - 2\hat{j} + 6\hat{k}$$

$|\vec{u}| = \sqrt{9+4+36} = \sqrt{49} = 7$

$\hat{u} = \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$

$\vec{\nabla} f = (y+z)\hat{i} + (x+z)\hat{j} + (y+x)\hat{k}$

$\vec{\nabla} f|_{P} = (3-1)\hat{i} + (2-1)\hat{j} + (3+2)\hat{k} = 2\hat{i} + \hat{j} + 5\hat{k}$

$D_u f = \vec{\nabla} f|_{P_0} \cdot \hat{u} = (2\hat{i} + \hat{j} + 5\hat{k}) \cdot (\frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k})$

$= \frac{6}{7} - \frac{2}{7} + \frac{30}{7} = \frac{34}{7}$

(9 points) *14.* Given $f(x,y)=3y^2-2y^3-3x^2+6xy$. Find all local maxima, local minima, and saddle points.

$f_x=-6x+6y$

$-6x+6y=0$

$y=x$

$f_y=6y-6y^2+6x$

$6y-6y^2+6y=0$

$12y-6y^2=0$

$6y(2-y)=0$

$y=0,2$

$y=0 \quad x=0$

$y=2 \quad x=2$

$f_{xx}=-6 \quad f_{yy}=6-12y \quad f_{xy}=6$

$D=f_{xx}f_{yy}-f_{xy}^2=(-6)(6-12y)-(6)^2$

$(0,0)$: $D=(-6)(6-12(0))-(6)^2=-36-36=-72<0$ saddle point

$(2,2)$: $D=(-6)(6-12(2))-(6)^2=(-6)(-18)-36=72>0 \quad f_{xx}=-6<0$ local max

$(0,0)$ saddle point $\quad (2,2)$ local maximum.

(9 points) *15.* Use Lagrange Multipliers to find three positive numbers whose sum is 12 and whose product is a maximum.

$f(x,y,z)=xyz \qquad g(x,y,z)=x+y+z=12$

$\vec\nabla f=yz\,\hat\imath+xz\,\hat\jmath+xy\,\hat k \qquad \vec\nabla g=\hat\imath+\hat\jmath+\hat k$

$\vec\nabla f=\lambda\vec\nabla g$

$yz=\lambda \quad xz=\lambda \quad xy=\lambda \quad x+y+z=12$

$yz=xz \qquad xy=xz \qquad \Rightarrow \quad y+y+y=12$

$z\neq0$: $y=x \qquad x\neq0$: $y=z$

$3y=12$

$y=4$

$x=4$

$z=4$

Optional Extra Credit Question:

(4 points) 16. Use the limit definition to find the directional derivative of $f(x,y)=2x^2+y^2$ at the point $P_0(1,-1)$ in the direction of $\vec{u}=-3\hat{i}+4\hat{j}$ $|\vec{u}|=5$

x_0 y_0

$\hat{u}=-\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}$

a b

$$\lim_{h\to 0}\frac{f(x+ha,y+hb)-f(x,y)}{h}$$

$$\lim_{h\to 0}\frac{f(1-\frac{3}{5}h,-1+\frac{4}{5}h)-f(1,-1)}{h}$$

$$=\lim_{h\to 0}\frac{\left[2(1-\frac{3}{5}h)^2+(-1+\frac{4}{5}h)^2\right]-\left[2(1)^2+(-1)^2\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left[2(1-\frac{6}{5}h+\frac{9}{25}h^2)+(1-\frac{8}{5}h+\frac{16}{25}h^2)\right]-[3]}{h}$$

$$=\lim_{h\to 0}\frac{2-\frac{12}{5}h+\frac{18}{25}h^2+1-\frac{8}{5}h+\frac{16}{25}h^2-3}{h}$$

$$=\lim_{h\to 0}\frac{-4h+\frac{34}{25}h^2}{h}=\lim_{h\to 0}\frac{h(-4+\frac{34}{25}h)}{h}$$

$$=\lim_{h\to 0}(-4+\frac{34}{25}h)=-4$$