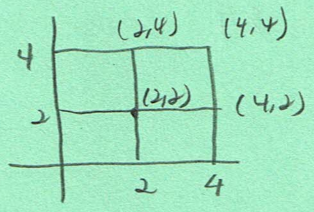
MATH 280 – EXAM #3
Fall Semester 2018

Name: KEY

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Remember, this exam is to show what you know. You may not use any notes or the textbook. You may not use any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam. You may not use a graphing calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(6 points) 1. Estimate $\iint\limits_R \left(x+x^2y\right)dA$ where $R=[0,4]\times[0,4]$ by dividing R into four equal squares and evaluating the function at the upper right corner of each square.



$\Delta A = 4$

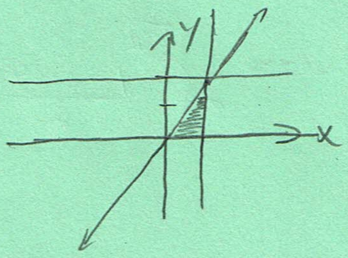
$[2+2^2(2)](4)+[4+(4)^2(2)](4)+[(2)+(2)^2(4)](4)+[(4)+(4)^2(4)](4)$

$= 40+144+72+272$

$= 528$

(4 points) 2. Sketch the region of integration and reverse the order of integration.

$$\int_0^2\int_{y/2}^1 e^{x^2}\,dx\,dy$$



$x=\frac{y}{2}$

$2x=y$

$$\int_0^1\int_0^{2x} e^{x^2}\,dy\,dx$$

(10 points) *3.* Evaluate the double integral $\iint\limits_D \left(x^2+2y\right)dA$ where D is the region bounded by $y=x$, $y=x^3$, $x\geq 0$.

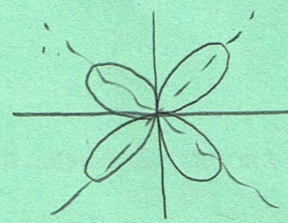
$$\int_0^1\int_{x^3}^{x}(x^2+2y)\,dy\,dx = \int_0^1 (x^2y+y^2)\Big|_{x^3}^{x}\,dy\,dx$$

$$=\int_0^1 \left[\left[x^2(x)+(x)^2\right]-\left[x^2(x^3)+(x^3)^2\right]\right]dx = \int_0^1 (x^3+x^2-x^5-x^6)\,dx$$

$$=\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^6}{6}-\frac{x^7}{7}\Big|_0^1 = \frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{7} = \frac{21+28-14-12}{84} = \frac{23}{84}$$

(7 points) *4.* Use a double integral to find the area of the region enclosed by one leaf of the graph $r=\sin 2\theta$.

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	0	1	0	-1	0	1	0	-1	0



$\sin 2\theta = 0$

$2\theta = 0$

$\theta = 0, \pi/2$

$$\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \int_0^{\pi/2}\frac{r^2}{2}\Big|_0^{\sin 2\theta}d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{1-\cos 4\theta}{2}\,d\theta$$

$$=\frac{1}{2}\left[\frac{1}{2}\theta-\frac{\sin 4\theta}{8}\right]\Big|_0^{\pi/2} = \frac{\pi}{8}$$

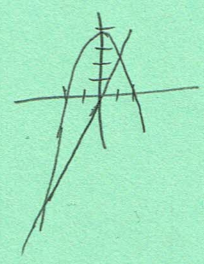
(10 points) 5. Find the volume of the solid whose base is the region in the *xy*-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 6$.

$4 - x^2 = 3x$

$0 = x^2 + 3x - 4$

$0 = (x+4)(x-1)$

$x = -4, 1$



$$\int_{-4}^{1}\int_{3x}^{4-x^2}\int_{0}^{x+6} dz\,dy\,dx = \int_{-4}^{1}\int_{3x}^{4-x^2} z\Big|_{0}^{x+6} dy\,dx = \int_{-4}^{1}\int_{3x}^{4-x^2} (x+6)\,dy\,dx$$

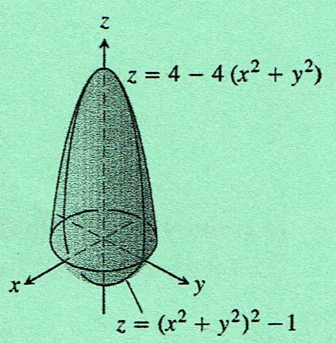
$$= \int_{-4}^{1} (xy + 6y)\Big|_{3x}^{4-x^2} dx = \int_{-4}^{1}\Big[\big[x(4-x^2) + 6(4-x^2)\big] - \big[x(3x) + 6(3x)\big]\Big]dx$$

$$= \int_{-4}^{1} (4x - x^3 + 24 - 6x^2 - 3x^2 - 18x)\,dx = \int_{-4}^{1} (-x^3 - 9x^2 - 14x + 24)\,dx$$

$$= \left(-\frac{x^4}{4} - 3x^3 - 7x^2 + 24x\right)\Big|_{-4}^{1} = \left[\left(-\frac{1}{4} - 3 - 7 + 24\right) - (-64 + 192 - 112 - 96)\right]$$

$$= 94 - \frac{1}{4} = \frac{376 - 1}{4} = \frac{375}{4}$$

(10 points) 6. Find the volume of the solid bounded above by $z = 4 - 4\left(x^2 + y^2\right)$ and bounded below by $z = \left(x^2 + y^2\right)^2 - 1$. Use cylindrical coordinates.



when $z = 0$:

$4 - 4r^2 = 0$

$1 = r^2$

$r = 1$

$$\int_{0}^{2\pi}\int_{0}^{1}\int_{r^4-1}^{4-4r^2} r\,dz\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{1} rz\Big|_{r^4-1}^{4-4r^2} dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{1} r\left[(4 - 4r^2) - (r^4 - 1)\right] dr\,d\theta = \int_{0}^{2\pi} r(4 - 4r^2 - r^4 + 1)\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{1} (5r - 4r^3 - r^5)\,dr\,d\theta = \int_{0}^{2\pi}\left(\frac{5}{2}r^2 - r^4 - \frac{r^6}{6}\right)\Big|_{0}^{1} d\theta = \int_{0}^{2\pi}\left(\frac{5}{2} - 1 - \frac{1}{6}\right) d\theta$$

$$= \int_{0}^{2\pi}\left(\frac{15 - 6 - 1}{6}\right) d\theta = \frac{4}{3}\int_{0}^{2\pi} d\theta = \frac{4}{3}\theta\Big|_{0}^{2\pi} = \frac{8\pi}{3}$$

(8 points) 7. Find the volume of the solid that lies above the cone $\varphi = \pi/3$ and below the sphere $\rho = 1$.

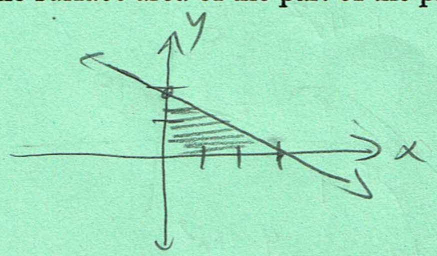
$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left(\frac{\rho^3}{3}\sin\varphi\right)\Big|_0^1 d\varphi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \sin\varphi \, d\varphi \, d\theta = \frac{1}{3}\int_0^{2\pi}(-\cos\varphi)\Big|_0^{\pi/3} d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left(-\frac{1}{2}+1\right) d\theta = \frac{1}{6}\int_0^{2\pi} d\theta = \frac{1}{6}\theta\Big|_0^{2\pi} = \pi/3$$

(5 points) 8. Set up the integral that would compute the surface area of the part of the plane $2x + 3y + z = 6$ that lies in the first octant.

$f(x,y) = z = 6 - 2x - 3y$



$2x + 3y = 6$

$3y = -2x + 6$

$y = -\frac{2}{3}x + 2$

$$S.A. = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dA$$

$$S.A. = \int_0^3 \int_0^{-\frac{2}{3}x+2} \sqrt{4+9+1} \, dy \, dx = \int_0^3 \int_0^{-\frac{2}{3}x+2} \sqrt{13} \, dy \, dx$$

(8 points) 9. Given $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $0 \le t \le 2$. Find the work done by $\vec{F}$ over $\vec{r}(t)$.

$$\int_0^2 \vec{F} \cdot \vec{r}\,'(t)\, dt = \int_0^2 [(t^2\hat{i} + t^3\hat{j} + t\hat{k}) \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k})]\, dt$$

$$= \int_0^2 (t^2 + 2t^4 + 3t^3)\, dt = \left(\frac{t^3}{3} + \frac{2t^5}{5} + \frac{3t^4}{4}\right)\Big|_0^2$$

$$= \frac{8}{3} + \frac{64}{5} + 12 = \frac{40 + 192 + 180}{15} = \frac{412}{15}$$

(8 points) 10. Evaluate $\int_C (3x + y - 2z)\,ds$ where C is the straight line segment from $(1,-3,2)$ to $(3,2,-1)$

$$x = 1 + 2t, \quad y = -3 + 5t, \quad z = 2 - 3t$$

$$\int_C (3x + y - 2z)\,ds = \int_0^1 (3(1+2t) + (-3+5t) - 2(2-3t))\sqrt{4+25+9}\, dt$$

$$= \sqrt{38}\int_0^1 (17t - 4)\, dt = \sqrt{38}\left(\frac{17t^2}{2} - 4t\right)\Big|_0^1 = \sqrt{38}\left(\frac{17}{2} - 4\right)$$

$$= \frac{9\sqrt{38}}{2}$$

11. Given the transformation $u = x + 2y$, $v = x - y$.

(3 points) a. Solve the system for x and y in terms of u and v.

$x = u - 2y$

$v = u - 2y - y$

$3y = u - v$

$y = \frac{1}{3}u - \frac{1}{3}v$

$x = u - 2\left(\frac{1}{3}u - \frac{1}{3}v\right)$

$x = u - \frac{2}{3}u + \frac{2}{3}v$

$x = \frac{1}{3}u + \frac{2}{3}v$

(3 points) b. Find the Jacobian, $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

(5 points) c. Find the image under the given transformation of the triangular region bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$.

$y = 0 \Rightarrow \frac{1}{3}u - \frac{1}{3}v = 0 \Rightarrow u = v$

$y = x \Rightarrow \frac{1}{3}u - \frac{1}{3}v = \frac{1}{3}u + \frac{2}{3}v \Rightarrow -\frac{1}{3}v = \frac{2}{3}v$

$\Rightarrow v = 0$

$x + 2y = 2 \Rightarrow \frac{1}{3}u + \frac{2}{3}v + 2\left(\frac{1}{3}u - \frac{1}{3}v\right) = 2$

$\Rightarrow \frac{1}{3}u + \frac{2}{3}v + \frac{2}{3}u - \frac{2}{3}v = 2$

$u = 2$

(3 points) *12.* Given $\vec{F} = (8xz)\hat{i} + (1-6yz^3)\hat{j} + (4x^2 - 9y^2z^2)\hat{k}$. Verify that $\vec{F}$ is conservative.

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = 8x = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = -18yz^2 = \frac{\partial R}{\partial y}$$

(8 points) *13.* Given $\vec{F}(x,y,z) = (-yz + 2x\ln y)\hat{i} + \left(\frac{x^2}{y} - xz\right)\hat{j} + (-xy)\hat{k}$ is a conservative vector field. Find the potential function.

$$\int (-yz + 2x\ln y)\,dx = -xyz + x^2\ln y + g(y,z) = f(x,y,z)$$

$$\frac{\partial f}{\partial y} = -xz + \frac{x^2}{y} + \frac{\partial g}{\partial y} = \frac{x^2}{y} - xz = Q$$

$$\frac{\partial g}{\partial y} = 0 \quad \Rightarrow \quad g(y,z) = h(z)$$

$$f(x,y,z) = -xyz + x^2\ln y + h(z)$$

$$\frac{\partial f}{\partial z} = -xy + h'(z) = -xy$$

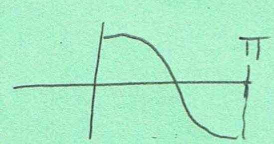
$$h'(z) = 0$$

$$h(z) = C$$

$$f(x,y,z) = -xyz + x^2\ln y + C$$

14. Given a thin plate of constant density $\delta = 3$ bounded by the x-axis and $y = \cos x$, $0 \le x \le \pi$.

(2 points) *a.* Set up the integral that would compute M_x.



$$\int_0^{\pi/2}\int_0^{\cos x} 3y\,dy\,dx + \int_{\pi/2}^{\pi}\int_{\cos x}^{0} 3y\,dy\,dx$$

(2 points) *b.* Set up the integral that would compute I_y.

$$\int_0^{\pi/2}\int_0^{\cos x} 3x^2\,dy\,dx + \int_{\pi/2}^{\pi}\int_{\cos x}^{0} 3x^2\,dy\,dx$$