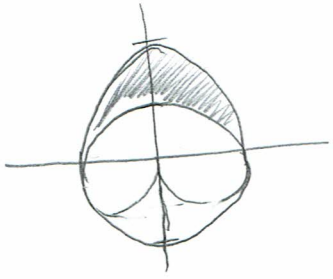
MATH 280 – QUIZ #5 *Name:* KEY

Directions: Please show all work for maximum credit. This quiz is worth 16 points. Good luck!

(5 points) 1. Find the area of the region that lies outside the circle $r = 1$ and inside the cardioid $r = 1 + \sin\theta$.



$1 + \sin\theta = 1$

$\sin\theta = 0$

$\theta = 0, \pi$

$$\int_0^{\pi}\int_1^{1+\sin\theta} r\,dr\,d\theta = \int_0^{\pi} \frac{r^2}{2}\Big|_1^{1+\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} ((1+\sin\theta)^2 - 1^2)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (1 + 2\sin\theta + \sin^2\theta - 1)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (2\sin\theta + \sin^2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi}\left(2\sin\theta + \frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[-2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi} = \frac{1}{2}\left[2 + \frac{\pi}{2} + 2\right] = 2 + \frac{\pi}{4}$$

(3 points) 2. Evaluate the following integral: $\int_0^2 \int_0^{2-x} \int_0^{2-x-y} dz\,dy\,dx$

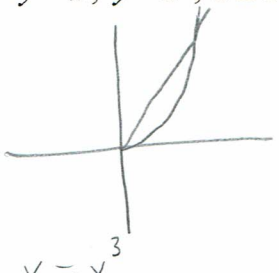
$$= \int_0^2 \int_0^{2-x} z\Big|_0^{2-x-y} dy\,dx = \int_0^2 \int_0^{2-x} (2-x-y)\,dy\,dx = \int_0^2 \left(2y - xy - \frac{y^2}{2}\right)\Big|_0^{2-x} dx$$

$$= \int_0^2 \left(2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right) dx = \int_0^2 \left(4 - 2x - 2x + x^2 - \frac{4-4x+x^2}{2}\right) dx$$

$$= \int_0^2 \left(4 - 4x + x^2 - 2 + 2x - \frac{1}{2}x^2\right) dx = \int_0^2 \left(2 - 2x + \frac{1}{2}x^2\right) dx$$

$$= \left(2x - x^2 + \frac{1}{6}x^3\right)\Big|_0^2 = 4 - 4 + \frac{4}{3} = \frac{4}{3}$$

(5 points) *3.* Evaluate the double integral $\iint_D (x^2+2y)\,dA$ where D is the region bounded by $y=x$, $y=x^3$, $x\geq 0$.



$x = x^3$

$0 = x^3 - x$

$0 = x(x^2-1)$

$0 = x(x+1)(x-1)$

$x = 0, 1, -1$

$$\int_0^1 \int_{x^3}^{x} (x^2+2y)\,dy\,dx = \int_0^1 (x^2y + y^2)\Big|_{x^3}^{x}\,dx$$

$$= \int_0^1 \left[(x^3+x^2) - (x^5+x^6)\right] dx$$

$$= \int_0^1 (x^3+x^2-x^5-x^6)\,dx$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{3} - \frac{x^6}{6} - \frac{x^7}{7}\right)\Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7}$$

$$= \frac{21+28-14-12}{84} = \frac{23}{84}$$

(3 points) *4.* Find the volume of the region cut from the solid sphere $\rho \leq 3$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.

$$\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/6} \int_0^{\pi/2} \frac{\rho^3}{3}\sin\phi\Big|_0^3\,d\phi\,d\theta$$

$$= 9\int_0^{\pi/6} \int_0^{\pi/2} \sin\phi\,d\phi\,d\theta = 9\int_0^{\pi/6} (-\cos\phi)\Big|_0^{\pi/2}\,d\theta$$

$$= 9\int_0^{\pi/6} (0+1)\,d\theta = 9\theta\Big|_0^{\pi/6} = 3\pi/2$$