MATH 280 – QUIZ #5 *Name:* KEY

Directions: Please show all work for maximum credit. This quiz is worth 14 points. Good luck!

(3 points) 1. Find the volume of the region cut from the solid sphere $\rho \le 3$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.

$$\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/6}\int_0^{\pi/2} \frac{\rho^3}{3} \sin\phi \Big|_0^3 \, d\phi \, d\theta$$

$$= 9\int_0^{\pi/6}\int_0^{\pi/2} \sin\phi \, d\phi \, d\theta = 9\int_0^{\pi/6} (-\cos\phi)\Big|_0^{\pi/2} d\theta = 9\int_0^{\pi/6} d\theta$$

$$= 9\theta \Big|_0^{\pi/6} = 9\left(\frac{\pi}{6}\right) = \frac{3\pi}{2}$$

(3 points) 2. Given $u = x + 2y$ and $v = x - y$. Find the Jacobian, $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$u = x + 2(x - v)$

$u = x + 2x - 2v$

$u = 3x - 2v$

$3x = u + 2v$

$x = \frac{1}{3}u + \frac{2}{3}v$

$y = x - v$

$y = \frac{1}{3}u + \frac{2}{3}v - v$

$y = \frac{1}{3}u - \frac{1}{3}v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$

$$= -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

(3 points) 3. Evaluate $\int_C (x - y + z - 2)\,ds$ where C is the straight-line segment from $(0,0,1)$ to $(2,1,0)$.

$x = 2t,\ y = t,\ z = 1 - t \quad , \quad 0 \le t \le 1$

$$\int_C (x - y + z - 2)\,ds = \int_0^1 (2t - t + 1 - t - 2)\sqrt{(2)^2 + (1)^2 + (-1)^2}\,dt$$

$$= \int_0^1 -\sqrt{6}\,dt = -\sqrt{6}\,t\Big|_0^1 = -\sqrt{6}$$

(3 points) 4. Find the work done by the field $\vec{F} = y\hat{i} + x\hat{j}$ around the circle $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}$, $0 \le t \le 2\pi$.

$$\int_0^{2\pi} \vec{F}(t) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \left[(\sin t\,\hat{i} + \cos t\,\hat{j}) \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j})\right] dt$$

$$= \int_0^{2\pi} (-\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} \cos 2t\,dt = \frac{\sin 2t}{2}\Big|_0^{2\pi} = 0$$

(4 points) 5. Given that the following field $\bar{F} = 2xy\hat{i} + \left(x^2 - z^2\right)\hat{j} - 2yz\hat{k}$ is conservative. Find a potential function f for the field.

$$\frac{\partial f}{\partial x} = P = 2xy$$

$$f(x,y,z) = \int 2xy\,dx = x^2y + g(y,z)$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2 = Q$$

$$\frac{\partial g}{\partial y} = -z^2$$

$$g(y,z) = -yz^2 + h(z)$$

$$f(x,y,z) = x^2y - yz^2 + h(z)$$

$$\frac{\partial f}{\partial z} = -2yz + h'(z) = -2yz = R$$

$$h'(z) = 0$$

$$h(z) = C$$

$$f(x,y,z) = x^2y - yz^2 + C$$