MATH 290 – EXAM #1 **_Name:_** Key

Spring Semester 2019

Directions: **_Please show all work for maximum credit._** _No work = no credit._ Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may not use a calculator on this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You will need your Mt. SAC student identification card to submit this exam. Good luck!

_1._ For each of the following differential equations, state the order and whether the equation is linear or nonlinear.

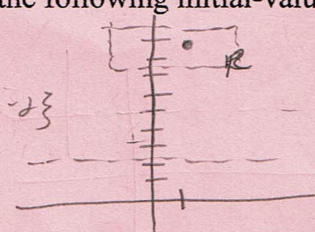
(2 points) a. $\left(1+x^2\right)y''+\left(1+y^4\right)y'=x^3$ 2nd order, nonlinear

(2 points) b. $4xy^{(5)}-2x^3y'''+7xy=3-\ln x$ 5th order, linear

(2 points) c. $(\cos x)y'''-e^xy''+\dfrac{y}{1+y}=\sec x$ 3rd order, nonlinear

_2._ Use the existence and uniqueness theorem to determine if each of the following initial-value problems has a unique solution, if possible.

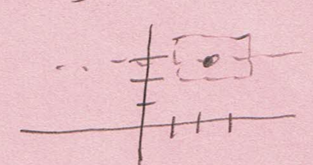
(3 points) a. $y'=\sqrt{y^2-4}, \quad y(1)=9$ Domain f: $\{y \mid y\geq 2, y\leq -2\}$



$\dfrac{\partial f}{\partial y}=\dfrac{2y}{2(y^2-4)^{1/2}}$ ← undefined at $y=\pm 2$ and defined for $y>2, y<-2$

So, a rectangle R can be defined around $(1,9)$ in which $f(x,y)$ and $\frac{\partial f}{\partial y}$ is defined. Therefore, there is a unique solution to the DE at $(1,9)$.

(3 points) b. $y'=\sqrt{9-y^2}, \quad y(2)=3$ Domain f: $\{y \mid -3\leq y\leq 3\}$



$\dfrac{\partial f}{\partial y}=\dfrac{-2y}{2(9-y)^{1/2}}$ ← undefined at $y=\pm 3$ defined for $-3<y<3$

It is not possible to define a rectangle around $(2,3)$ in which $\frac{\partial f}{\partial y}$ is defined. Therefore, the existence and uniqueness theorem is inconclusive.

(4 points) *3.* Verify that $y(x) = 2x^2 - 1 + ce^{-2x^2}$ is a solution to the differential equation

$\dfrac{dy}{dx} + 4xy = 8x^3$

$y'(x) = 4x + c\,e^{-2x^2}(-4x)$

$4x - 4xce^{-2x^2} + 4x(2x^2 - 1 + ce^{-2x^2}) = 4x - 4xce^{-2x^2} + 8x^3 - 4x + 4xce^{-2x^2}$

$= 8x^3$

4. Solve the following differential equations.

(10 points) *a.* $x\dfrac{dy}{dx} - (1+x)y = xy^2$

$\dfrac{dy}{dx} - \dfrac{1+x}{x}y = y^2$

$y^{-2}\dfrac{dy}{dx} - \dfrac{1+x}{x}y^{-1} = 1$

$u = y^{-1}$

$\dfrac{du}{dx} = -y^{-2}\dfrac{dy}{dx}$

$-\dfrac{du}{dx} = y^{-2}\dfrac{dy}{dx}$

$-\dfrac{du}{dx} - \dfrac{1+x}{x}u = 1$

$\dfrac{du}{dx} + \dfrac{1+x}{x}u = -1$

$\mu(x) = e^{\int \frac{1+x}{x}dx} = e^{\int(\frac{1}{x}+1)dx}$

$= e^{(\ln|x| + x)} = |x|e^x$

$\dfrac{d}{dx}(xe^x u) = -xe^x$

$xe^x u = -\int xe^x dx$

$xe^x u = -[xe^x - e^x + c]$

$xe^x y^{-1} = -xe^x + e^x + C_1$

(10 points) *b.* $(x^2-y^2)dx+(y^2-xy)dy=0$

$M(tx,ty) = t^2x^2 - t^2y^2 = t^2(x^2-y^2) = t^2M(x,y)$ $\therefore$ homogeneous of degree 2.

$N(tx,ty) = t^2y^2 - tx\,ty = t^2(y^2-xy) = t^2N(x,y)$

let $u=\frac{y}{x}$ or $y=ux$

so, $dy = u\,dx + x\,du$

$(x^2-u^2x^2)dx + (u^2x^2 - xux)(u\,dx + x\,du) = 0$

$x^2dx - u^2x^2dx + u^3x^2dx + u^2x^3du - u^2x^2dx - x^3u\,du = 0$

$(x^2+u^3x^2-2u^2x^2)dx + (u^2x^3 - ux^3)du = 0$

$x^2(1+u^3-2u^2)\,dx = -x^3(u^2-u)\,du$

$-\frac{1}{x}\,dx = \frac{u^2-u}{u^3-2u^2+1}\,du$

$-\int\frac{1}{x}\,dx = \int\frac{u^2-u}{u^3-2u^2+1}\,du$

$-\ln|x| + C = \int\frac{u^2-u}{u^3-2u^2+1}\,du$ OK to stop here

(10 points) *c.* $(y^2\cos x - 3x^2y - 2x)dx + (2y\sin x - x^3 + \ln y)dy = 0$

$M_y = 2y\cos x - 3x^2$ $N_x = 2y\cos x - 3x^2$ $\therefore$ exact

$f(x,y) = \int M_x\,dx = \int(y^2\cos x - 3x^2y - 2x)dx = y^2\sin x - x^3y - x^2 + g(y)$

$\frac{\partial f}{\partial y} = 2y\sin x - x^3 + g'(y) = N(x,y)$

$2y\sin x - x^3 + g'(y) = 2y\sin x - x^3 + \ln y$

$g'(y) = \ln y$

$g(y) = \int \ln y\,dy$

$g(y) = y\ln y - y + C_1$

$f(x,y) = y^2\sin x - x^3y - x^2 + y\ln y - y + c_1$

solution: $\boxed{y^2\sin x - x^3y - x^2 + y\ln y - y = C}$

(10 points) *d.* $x^2\dfrac{dy}{dx}=y-xy$

$$x^2\frac{dy}{dx}=y(1-x)$$

$$\frac{1}{y}\,dy=\frac{1-x}{x^2}\,dx$$

$$\frac{1}{y}\,dy=\left(\frac{1}{x^2}-\frac{1}{x}\right)dx$$

$$\ln|y|=-\frac{1}{x}-\ln|x|+C$$

(10 points) *e.* $\left(1-x^2\right)\dfrac{dy}{dx}+2xy=4x\left(1-x^2\right),\quad -1<x<1$

$$\frac{dy}{dx}+\frac{2x}{1-x^2}y=4x$$

$$\mu(x)=e^{\int\frac{2x}{1-x^2}dx}=e^{-\ln|1-x^2|}=\frac{1}{1-x^2}$$

$$\frac{d}{dx}\left[\frac{1}{1-x^2}\,y\right]=\frac{4x}{1-x^2}$$

$$\left(\frac{1}{1-x^2}\right)y=\int\frac{4x}{1-x^2}\,dx$$

$$\frac{1}{1-x^2}\,y=-2\ln|1-x^2|+C$$

(10 points) 5. Solve the following initial-value problem.

$$\left(e^x + y\right)dx + \left(2 + x + ye^y\right)dy = 0, \quad y(0) = 1$$

$M_y = 1$ $N_x = 1$ $\therefore$ exact

$f(x,y) = \int (e^x + y)\,dx = e^x + xy + g(y)$

$\frac{\partial f}{\partial y} = x + g'(y) = N$

$x + g'(y) = 2 + x + ye^y$

$g'(y) = 2 + ye^y$

$g(y) = 2y + ye^y - e^y$

$e^x + xy + 2y + ye^y - e^y = C$

$y(0) = 1$ $e^0 + (0)(1) + 2(1) + 1e^1 - e^1 = C$ $e^x + xy + 2y + ye^y - e^y = 3$

$1 + 2 + e - e = C$

$C = 3$

6. Determine the integrating factor necessary to change the following differential equations into an exact equation. Do not solve the differential equation.

(5 points) a. $\left(y - x^2\right)dx + 2xdy = 0, \quad x > 0$

$M_y = 1$ $N_x = 2$

$\frac{M_y - N_x}{N} = \frac{1-2}{2x} = -\frac{1}{2x} = f(x)$

$\mu(x) = e^{\int -\frac{1}{2x}dx}$

$= e^{-\frac{1}{2}\ln|x|} = x^{-1/2}$

(5 points) b. $\cos x dx + \left(1 + \frac{2}{y}\right)\sin x dy = 0$

$My = 0$ $N_x = (1 + \frac{2}{y})\cos x$

$\frac{M_y - N_x}{M} = \frac{-(1+\frac{2}{y})\cos x}{\cos x} = -(1 + \frac{2}{y}) = f(y)$

$\mu(y) = e^{\int (1+\frac{2}{y})dy}$

$= e^{(y + 2\ln(y))}$

$= e^{y + 2\ln|y|}$

$= e^{y + \ln|y|^2} = y^2 e^y$

7. Given the following differential equation: $y' = y^2 - 6y + 5$

(2 points) *a.* Determine all equilibrium solutions.

$(y-5)(y-1) = 0$

$y = 5, \; y = 1$

(4 points) *b.* Determine the regions in the *xy*-plane where the solutions are increasing, and determine the regions where they are decreasing.

y'

$y' > 0$ increasing

5

$y' < 0$ decreasing

1

$y' > 0$ increasing

increasing $(-\infty, 1) \cup (5, \infty)$

decreasing $(1, 5)$

(4 points) *c.* Determine the regions in the *xy*-plane where the solutions curves are concave up, and determine the regions where they are concave down.

$y'' = (2y-6)y' = 2(y-3)(y-5)(y-1)$

y''

$y'' > 0$ concave up

5

$y'' < 0$ concave down

3

$y'' > 0$ concave up

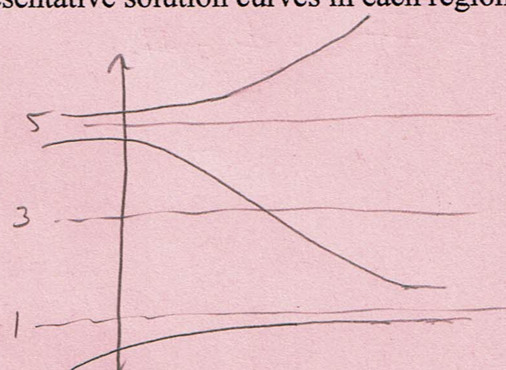
1

$y'' < 0$ concave down

concave up $(1,3) \cup (5, \infty)$

concave down $(-\infty, 1) \cup (3, 5)$

(4 points) *d.* Sketch representative solution curves in each region determined by parts *a, b* and *c.*



(2 points) *e.* Classify the equilibrium solutions as stable, unstable, or semi-stable.

$y = 1$ stable

$y = 5$ unstable.