**MATH 290 – EXAM #1**
**Summer Session 2019**

**Name:** KEY

**Directions:** ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You will need your Mt. SAC student identification card to submit this exam. Good luck!

*1.* For each of the following differential equations, state the order and whether the equation is linear or nonlinear.

*(2 points)* *a.* $x^2y^{(4)}+\left(1+x^3\right)y'=y\sin x$ 4th order linear

*(2 points)* *b.* $2x^2y^5+4x^3y'''+7xy=3$ 3rd order nonlinear

*(2 points)* *c.* $e^xy''+(\ln x)y'+\dfrac{y}{1+x}=\sec x$ 2nd order linear

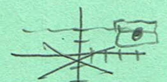
*2.* Use the existence and uniqueness theorem to determine if each of the following initial-value problems has a unique solution, if possible.

*(3 points)* *a.* $y'=\sqrt{y^2-4},\quad y(4)=2$

$y^2-4>0$
$y^2>4$
$|y|>2$

$$\frac{\partial f}{\partial y}=\frac{y}{(y^2-4)^{1/2}}$$

$\frac{\partial f}{\partial y}$ is undefined at $(4,2)$. Thus, there is no rectangle around $(4,2)$ where $f(x,y)$ and $\frac{\partial f}{\partial y}$ is continuous.



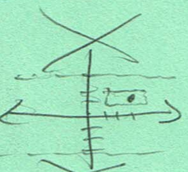
$\therefore$ The theorem is inconclusive.

*(3 points)* *b.* $y'=\sqrt{9-y^2},\quad y(3)=2$

$$\frac{\partial f}{\partial y}=\frac{-y}{\sqrt{9-y^2}}$$

$f(x,y)$ and $\frac{\partial f}{\partial y}$ are defined at $(3,2)$.
There is a rectangle around $(3,2)$ where $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous. $\therefore$ There exists a unique solution

$9-y^2>0$
$y^2<9$
$|y|<3$



*(4 points)* *3.* Given that $y = c_1x^4 + c_2x^2$ is a solution to a first-order differential equation. Find a solution to the corresponding initial-value problem given the initial condition of $y(2) = 32$, $y'(2) = 0$ .

$y = c_1x^4 + c_2x^2$

$y' = c_1 4x^3 + c_2 2x$

$-1(16c_1 + 4c_2 = 32)$

$32c_1 + 4c_2 = 0$

$16c_1 = -32$

$c_1 = -2$

$-64 + 4c_2 = 0$

$4c_2 = 64$

$c_2 = 16$

$y = -2x^4 + 16x^2$

*4.* Solve the following differential equations.

*(10 points)* *a.* $(2xy + \cos y)dx + (x^2 - x\sin y - 2y)dy = 0$

$\frac{\partial M}{\partial y} = 2x - \sin y \quad \frac{\partial N}{\partial x} = 2x - \sin y \quad \therefore$ exact

$f(x,y) = \int M dx = \int (2xy + \cos y)dx = x^2y + x\cos y + g(y)$

$\frac{\partial f}{\partial y} = x^2 - x\sin y + g'(y) = x^2 - x\sin y - 2y = N(x,y)$

$g'(y) = -2y$

$g(y) = -y^2 + C_1$

$f(x,y) = x^2y + x\cos y - y^2 + C_1$

Solution: $x^2y + x\cos y - y^2 = C$

*(10 points)* b. $e^x y \dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

$$e^x y \frac{dy}{dx} = e^{-y} + e^{-2x}e^{-y}$$

$$e^x y \frac{dy}{dx} = e^{-y}(1+e^{-2x})$$

$$y e^y dy = e^{-x}(1+e^{-2x})dx$$

$$y e^y dy = (e^{-x}+e^{-3x})dx$$

$$(ye^y - e^y) = -e^{-x} - \frac{1}{3}e^{-3x} + C$$

*(10 points)* c. $(x^2+2y^2)dx - (xy)dy = 0$

$M(x,y) = x^2+2y^2$ $N(x,y) = -(xy)$

$M(tx,ty) = (tx)^2 + 2(ty)^2 = t^2x^2 + 2t^2y^2 = t(x^2+2y^2)$ $N(tx,ty) = -(txty) = -t^2xy = t^2(-xy)$

$\therefore$ homogeneous of degree 2.

Let $u = y/x$. $y = ux \Rightarrow dy = u\,dx + x\,du$

$$(x^2+2(ux)^2)dx - (xux)(u\,dx + x\,du) = 0$$

$$(x^2+2u^2x^2)dx - ux^2(u\,dx + x\,du) = 0$$

$$(x^2+2u^2x^2)dx - u^2x^2dx - ux^3du = 0$$

$$(x^2+u^2x^2)dx = ux^3du$$

$$x^2(1+u^2)dx = ux^3du$$

$$\frac{1}{x}dx = \frac{u}{1+u^2}du$$

$$v = 1+u^2$$

$$dv = 2u\,du$$

$$\frac{1}{2}dv = u\,du$$

$$\frac{1}{x}dx = \frac{1}{2}\cdot\frac{1}{v}dv$$

$$\int \frac{1}{x}dx = \frac{1}{2}\int\frac{1}{v}dv$$

$$\ln|x| = \frac{1}{2}\ln|v| + C$$

$$\ln|x| = \frac{1}{2}\ln|1+u^2| + C$$

$$\ln|x| = \frac{1}{2}\ln|1+(\tfrac{y}{x})^2| + C$$

*(10 points)* *d.* $(\sin x)y' - y\cos x = \sin 2x$ First-order linear

$$(\sin x)y' - y\cos x = 2\sin x\cos x$$

$$y' - y\cot x = 2\cos x$$

$$\mu(x) = e^{-\int \cot x\, dx} = e^{-\ln|\sin x|} = \frac{1}{\sin x}$$

$$\frac{d}{dx}\left[\left(\frac{1}{\sin x}\right)y\right] = \frac{2\cos x}{\sin x}$$

$$(\csc x)y = \int 2\cot x\, dx$$

$$(\csc x)y = 2\ln|\sin x| + C$$

*(10 points)* *e.* $\dfrac{dy}{dx} + \dfrac{2e^x}{1+e^x}y = 2e^{-x}\sqrt{y} = 2e^{-x}y^{1/2}$ Bernoulli

$$y^{-1/2}\frac{dy}{dx} + \frac{2e^x}{1+e^x}y^{1/2} = 2e^{-x}$$

$$u = y^{1/2}$$

$$\frac{du}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$$

$$2\frac{du}{dx} = y^{-1/2}\frac{dy}{dx}$$

$$2\frac{du}{dx} + \frac{2e^x}{1+e^x}u = 2e^{-x}$$

$$\frac{du}{dx} + \frac{e^x}{1+e^x}u = e^{-x}$$

$$\mu(x) = e^{\int \frac{e^x}{1+e^x}dx} = e^{\ln|1+e^x|} = 1+e^x$$

$$\frac{d}{dx}\left[(1+e^x)u\right] = (1+e^x)e^{-x}$$

$$(1+e^x)u = \int (e^{-x}+1)\, dx$$

$$(1+e^x)u = -e^{-x} + x + C$$

$$(1+e^x)y^{1/2} = -e^{-x} + x + C$$

*(10 points)* 5. Solve the following initial-value problem.

$$y' + (\tan x)\, y = \cos^2 x, \quad y(0) = -1$$

$\mu(x) = e^{\int \tan x\,dx} = e^{-\ln|\cos x|} = (\cos x)^{-1}$

$\frac{d}{dx}\left[\left(\frac{1}{\cos x}\right) y\right] = \cos x$

$(\sec x)\, y = \int \cos x\, dx$

$(\sec x)\, y = \sin x + C$

$(\sec 0)(-1) = \sin 0 + C$

$-1 = C$

$(\sec x)\, y = \sin x - 1$

6. Determine the integrating factor necessary to change the following differential equations into an exact equation. Then, verify that the integration factor makes the resulting differential equation exact.

*(4 points)* a. $(x^2 y)\,dx + y\left(x^3 + e^{-3y}\sin y\right)dy = 0$

$M_y = x^2 \quad N_x = 3x^2 y$

$\frac{M_y - N_x}{M} = \frac{x^2 - 3x^2 y}{x^2 y} = \frac{x^2(1-3y)}{x^2 y} = \frac{1-3y}{y}$

$\mu(y) = e^{-\int \frac{1-3y}{y}dy} = e^{-\ln|y| + 3y} = \frac{e^{3y}}{y}$

$(x^2 e^{3y})\,dx + (x^3 e^{3y} + \sin y)\,dy = 0$

$M_y = 3x^2 e^{3y} \quad N_x = 3x^2 e^{3y}$

$\therefore$ exact

*(4 points)* b. $(2x - y^2)\,dx + (xy)\,dy = 0$

$M_y = -2y \quad N_x = y$

$\frac{M_y - N_x}{N} = \frac{-2y - y}{xy} = -\frac{3}{x}$

$\mu(x) = e^{-\int \frac{3}{x}dx} = e^{-3\ln|x|} = x^{-3}$

$(2x^{-2} - y^2 x^{-3})\,dx + (x^{-2} y)\,dy = 0$

$M_y = -2y x^{-3} \quad N_x = -2x^{-3} y$

$\therefore$ exact.

7. Given the following differential equation: $y' = 6 + 5y - y^2$

*(1 point)* *a.* Determine all equilibrium solutions.

$y' = -(y^2 - 5y - 6)$

$= -(y-6)(y+1)$

$y = 6, y = -1$

*(3 points)* *b.* Determine the regions in the *xy*-plane where the solutions are increasing, and determine the regions where they are decreasing.

| | | $-1$ | | $6$ | |
|---|---|---|---|---|---|
| $-1$ | $-$ | | $-$ | | $-$ |
| $y-6$ | $-$ | | $-$ | | $+$ |
| $y+1$ | $-$ | | $+$ | | $+$ |
| | $-$ | | $+$ | | $-$ |

increasing on $(-1, 6)$

decreasing on $(-\infty, -1) \cup (6, \infty)$

*(3 points)* *c.* Determine the regions in the *xy*-plane where the solutions curves are concave up, and determine the regions where they are concave down.

$y'' = (5-2y)y'$

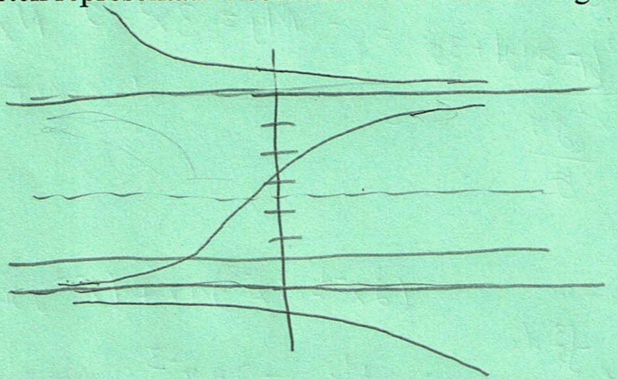
$= -(5-2y)(y-6)(y+1)$

| | | $-1$ | | $5/2$ | | $6$ | |
|---|---|---|---|---|---|---|---|
| $-1$ | $-$ | | $-$ | | $-$ | | $-$ |
| $5-2y$ | $+$ | | $+$ | | $-$ | | $-$ |
| $y-6$ | $-$ | | $-$ | | $-$ | | $+$ |
| $y+1$ | $-$ | | $+$ | | $+$ | | $+$ |
| | $-$ | | $+$ | | $-$ | | $+$ |

concave down on $(-\infty, -1) \cup (\frac{5}{2}, 6)$

concave up on $(-1, 5/2) \cup (6, \infty)$

*(3 points)* *d.* Sketch representative solution curves in each region determined by parts *a, b* and *c.*



*(2 points)* *e.* Classify the equilibrium solutions as stable, unstable, or semi-stable.

$y = -1$ is unstable

$y = 6$ is stable.

*(6 points)* *8.* Use Euler's method with the following function and given step size to find $y_1$, $y_2$, and $y_3$. Use at least four decimal places for your answers.

$$y' = xy^2 + y, \quad y(2) = 1, \quad h = 0.1$$

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + 0.1\left[2(1)^2 + 1\right] = 1 + 0.1(3) = 1.3$

$x_1 = 2.1 \quad y_1 = 1.3$

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 1.3 + 0.1\left[2.1(1.3)^2 + 1.3\right] = 1.3 + 0.1(4.849) = 1.7849$

$x_2 = 2.2 \quad y_2 = 1.7849$

$y_3 = y_2 + h\, f(x_2, y_2)$

$= 1.7849 + 0.1\left[2.2(1.7849)^2 + 1.7849\right]$

$= 1.7849 + 0.1(8.7938096222)$

$\approx 2.66428096222$

$x_3 = 2.3 \quad y_3 \approx 2.6643$