MATH 250 EXAM 1

1

$y' = \sqrt{1-y^2}$ $\quad \frac{\partial y}{\partial x} = \frac{-2y}{2(1-y^2)^{1/2}}$

$-1 < y < 1$

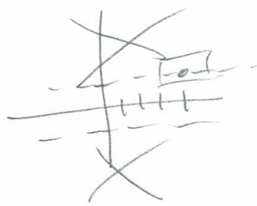
$y(1) = 3$

inconclusive

3

$y(4) = 1$

inconclusive

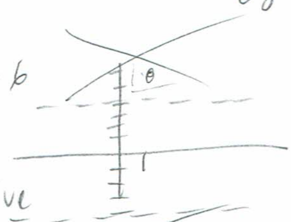


$y' = \sqrt{16-y^2}$ $\quad \frac{\partial f}{\partial y} = \frac{-2y}{2(1-y^2)^{1/2}}$

$-4 < y < 4$

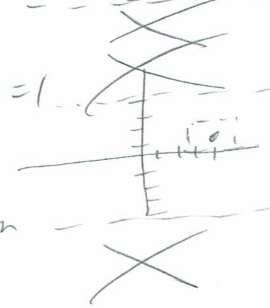
$y(1) = 6$

inconclusive



$y(4) = 1$

unique solution



2

$\frac{dy}{dx} - \frac{1}{2x\ln x} y = 2xy^3$

$y^{-3}\frac{dy}{dx} - \frac{1}{2x\ln x} y^{-2} = 2x$

$u = y^{-2}$

8

$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$

$-\frac{1}{2}\frac{du}{dx} = y^{-3}\frac{dy}{dx}$

$-\frac{1}{2}\frac{du}{dx} - \frac{1}{2x\ln x} u = 2x$

$\frac{du}{dx} + \frac{1}{x\ln x} u = -4x$

$\mu(x) = e^{\int \frac{1}{x\ln x}dx}$

$= e^{\ln|\ln x|} = \ln x$

$(\ln x)u = -4\int x\ln x\,dx$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}dx$

$\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$-4\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)$

$(\ln x)u = -2x^2\ln x + x^2 + C$

$(\ln x)y^{-2} = -2x^2\ln x + x^2 + C$

$\frac{dy}{dx} + \frac{2x}{1+x^2} y = xy^2$

$y^{-2}\frac{dy}{dx} + \frac{2x}{1+x^2} y^{-1} = x$

$u = y^{-1}$

$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{du}{dx} = y^{-2}\frac{dy}{dx}$

$-\frac{du}{dx} + \frac{2x}{1+x^2} u = x$

$\frac{du}{dx} - \frac{2x}{1+x^2} u = -x$

$\mu = e^{-\int \frac{2x}{1+x^2}dx}$

$= e^{-\ln|1+x^2|} = \frac{1}{1+x^2}$

$\frac{1}{1+x^2} u = -\int \frac{x}{1+x^2}dx$

$\frac{1}{1+x^2} y^{-1} = -\frac{1}{2}\ln|1+x^2| + C$

(3) $y' - y\tan x = 8\sin^3 x$

$\mu = e^{-\int \tan x\,dx} = e^{\ln|\cos x|} = \cos x$

$(\cos x)y = 8\int \sin^3 x \cos x\,dx$

8

$(\cos x)y = 2\sin^4 x + C$

$y' + y\cot x = 8\cos^3 x$

$\mu = e^{\int \cot x\,dx} = e^{\ln|\sin x|} = \sin x$

$(\sin x)y = 8\int \cos^3 x \sin x\,dx$

$(\sin x)y = -2\cos^4 x + C$

(4) $(y^2)dx + (x^2 + xy)dy = 0$

$M(tx,ty) = t^2y^2 = t^2M(x,y)$

$N(tx,ty) = t^2x^2 + txty = t^2(x^2+xy) = t^2N(x,y)$

$v = \frac{x}{y} \qquad x = vy$

$dx = v\,dy + y\,dv$

$y^2(v\,dy + y\,dv) + (v^2y^2 + vy^2)\,dy = 0$

$vy^2dy + y^3dv + v^2y^2dy + vy^2dy = 0$

$y^3dv + (v^2y^2 + 2vy^2)dy = 0$

8

$y^2(v^2+2v)dy = -y^3dv$

$-\frac{1}{y}\,dy = \frac{1}{v(v+2)}\,dv$

$\frac{1}{v(v+2)} = \frac{A}{v} + \frac{B}{v+2}$

$1 = A(v+2) + Bv$

$v = 0 \qquad A = 1/2$

$v = -2 \qquad B = -1/2$

$-\frac{1}{y}\,dy = \left(\frac{1/2}{v} + \frac{-1/2}{v+2}\right)dv$

$-\ln|y| = \frac{1}{2}\ln|v| - \frac{1}{2}\ln|v+2| + C$

$-\ln|y| = \frac{1}{2}\ln|\frac{x}{y}| - \frac{1}{2}\ln|\frac{x}{y}+2| + C$

⑤ $\frac{dy}{dx} = \frac{x^2(y^2+1)}{x^3+3}$ $\qquad$ $\frac{dy}{dx} = \frac{x(y^2+1)}{x^2+3}$

$\frac{dy}{y^2+1} = \frac{x^2}{x^3+3}\,dx$ $\qquad$ $\frac{dy}{y^2+1} = \frac{x}{x^2+3}\,dx$

6

$\tan^{-1} y = \frac{1}{3}\ln|x^3+3| + C$ $\qquad$ $\tan^{-1} y = \frac{1}{2}\ln|x^2+3| + C$

⑥ $[y^2\cos x - 3x^2y - 2x]\,dx + [2y\sin x - x^3 + \ln y]\,dy = 0$

$\frac{\partial M}{\partial y} = 2y\cos x - 3x^2$ $\qquad$ $\frac{\partial N}{\partial x} = 2y\cos x - 3x^2$ $\therefore$ exact

$\int M\,dx = \int (y^2\cos x - 3x^2y - 2x)\,dx = y^2\sin x - x^3y - x^2 + g(y) = f(x,y)$

$\frac{\partial f}{\partial y} = 2y\sin x - x^3 + g'(y) = N(x,y)$

8

$2y\sin x - x^3 + g'(y) = 2y\sin x - x^3 + \ln y$

$g'(y) = \ln y$

$g(y) = y\ln y - y$

$\int \ln y\,dy$

$u = \ln y \quad dv = dy$

$du = \frac{1}{y}dy \quad v = y$

$f(x,y) = y^2\sin x - x^3y - x^2 + y\ln y - y$

$y^2\sin x - x^3y - x^2 + y\ln y - y = C$

⑦ $x\frac{dy}{dx} + y = 6x+1 \quad y(1) = 7$ $\qquad$ $x\frac{dy}{dx} + y = 2x+1 \quad y(1) = 6$

8

$\frac{dy}{dx} + \frac{1}{x}y = \frac{6x+1}{x}$ $\qquad$ $\frac{dy}{dx} + \frac{1}{x}y = \frac{2x+1}{x}$

$\mu(x) = e^{\ln|x|} = x$ $\qquad$ $1\cdot 7 = 3\cdot 1 + 1 + C$ $\qquad$ $\mu(x) = e^{\ln|x|} = x$ $\qquad$ $1\cdot 6 = 1 + 1 + C$

$xy = \int (6x+1)\,dx$ $\qquad$ $7 = 4 + C$ $\qquad$ $xy = \int (2x+1)\,dx$ $\qquad$ $6 = 2 + C$

$C = 3$ $\qquad$ $4 = C$

$xy = 3x^2 + x + C$ $\qquad$ $xy = 3x^2 + x + 3$ $\qquad$ $xy = x^2 + x + C$ $\qquad$ $xy = x^2 + x + 4$

(8) $y' = xy^2 + y \quad y(2) = 1 \quad h = 0.1$

$y_1 = y_0 + f(x_0, y_0)h$

$y_1 = 1 + 3 \cdot 0.1 = 1.3$, $x_1 = 2.1$

$y_2 = 1.3 +$

5 $= 1.7849 \quad x_2 = 2.2$

$y_3 = 1.7849 +$

$= 2.6643$

$y' = xy^2 + x \quad y(1) = 2 \quad h = 0.1$

$y_1 = y_0 + f(x_0, y_0)h$

$= 2 + 0.5 = 2.5$, $x_1 = 1.1$

$y_2 = 2.5 +$

$= 3.2975 \quad x_2 = 1.2$

$y_3 = 3.2975 +$

$= 4.7223$

(9) $y' = 24 - 2y - y^2$

$-(y^2 + 2y - 24)$

2 $-(y+6)(y-4)$

$y = -6, \ y = 4$

$-$
4
$+$
-6
3 $-$

3

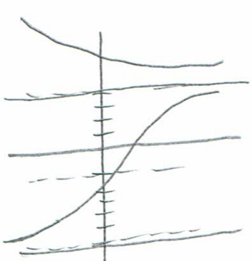
decreasing on $(-\infty, -6) \cup (4, \infty)$

increasing on $(-6, 4)$

$y'' = (-2 - 2y)y'$

$= -2(y+1)[-(y+6)(y-4)]$

$= 2(y+1)(y+6)(y-4)$



3 4 $+$
-1 $-$
$+$
-6 $-$

concave up $(-6, 1) \cup (4, \infty)$

concave down $(-\infty, -6) \cup (-1, 4)$

2 $y = -6$ unstable

$y = 4$ stable

$y' = 15 - 2y - y^2$

$-(y^2 + 2y - 15)$

$-(y+5)(y-3)$

$y = 3, \ y = -5$

$-$
3
$+$
-5
$-$

increasing on $(-5, 3)$

decreasing on $(-\infty, -5) \cup (3, \infty)$

$y'' = (-2 - 2y)y'$

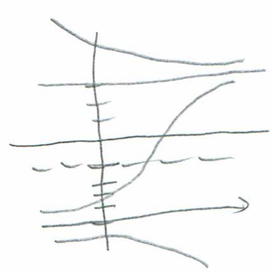
$= -2(y+1)[-(y+5)(y-3)]$

$= 2(y+1)(y+5)(y-3)$

3 $+$
$-$
-1
$+$
-5 $-$

concave up $(-5, -1) \cup (3, \infty)$

concave down $(-\infty, -5) \cup (-1, 3)$



$y = -5$ unstable

$y = 3$ stable

10

$(x^2 y)\,dx + y(x^3 + e^{-3y}\sin y)\,dy = 0$

$M_y = x^2 \qquad N_x = 3x^2 y$

$\dfrac{M_y - N_x}{N} = \times$

$\dfrac{N_x - M_y}{M} = \dfrac{3x^2 y - x^2}{x^2 y}$

$e^{\int (3 - \frac{1}{y})\,dy} = e^{3y - \ln|y|} = e^{3y} e^{-\ln|y|}$

$= y^{-1} e^{3y}$

4

$\dfrac{3x^2 y - x^2}{x^2 y} = \dfrac{3y - 1}{y} = 3 - \dfrac{1}{y}$

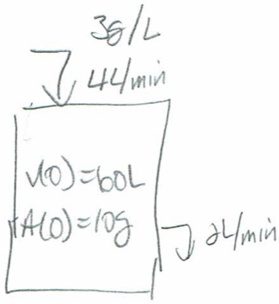
$y^{-1} e^{3y}(x^2 y)\,dx + y^{-1} e^{3y} y(x^3 + e^{-3y}\sin y)\,dy = 0$

$x^2 e^{3y}\,dx + (x^3 e^{3y} + \sin y)\,dy$

$M_y = 3x^2 e^{3y} \qquad N_x = 3x^2 e^{3y}$ ∴ exact

(11)

8



$$\frac{dV}{dt} = 4-2 = 2$$

$$V = 2t + C_1$$

$$V = 2t + 60$$

$$\frac{dA}{dt} = 12 - 2\frac{A}{V}$$

$$\frac{dA}{dt} = 12 - \frac{2}{2t+60}A$$

$$\frac{dA}{dt} + \frac{1}{t+30}A = 12$$

$$\mu(t) = e^{\int \frac{1}{t+30}dt} = t+30$$

$$(t+30)A = 12\int(t+30)\,dt$$

$$(t+30)A = 6(t+30)^2 + C_2$$

$$(30)(10) = 6(30)^2 + C_2$$

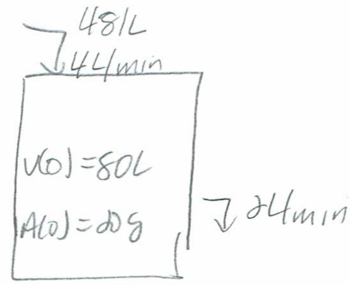
$$300 = 5400 + C_2$$

$$(t+30)A = 6(t+30)^2 - 5100$$

$$A = \frac{6(60)^2 - 5100}{60}$$

$$A = 275g$$

Concentration $\frac{275g}{100L} = 2.29\,g/L.$



$$\frac{dV}{dt} = 4-2 = 2$$

$$V = 2t + C_1$$

$$V = 2t + 80$$

$$\frac{dA}{dt} = 16 - 2\frac{A}{V}$$

$$\frac{dA}{dt} = 16 - \frac{2}{2t+80}A$$

$$\frac{dA}{dt} + \frac{1}{t+40}A = 16$$

$$\mu(t) = e^{\int \frac{1}{t+40}dt} = t+40$$

$$(t+40)A = 16\int(t+40)\,dt$$

$$(t+40)A = 8(t+40)^2 + C_2$$

$$(40)(20) = 8(40)^2 + C_2$$

$$800 = 12800 + C_2$$

$$(t+40)A = 8(t+40)^2 - 12000$$

$$A = \frac{8(70)^2 - 12000}{70}$$

$$A = 388.57\,g$$

Concentration $= \frac{388.57g}{140L} = 2.76\,g/L$

(12)

$T = T_m + Ce^{kt}$

$T = 300 + Ce^{kt}$ $\quad T(0)=50$

$T(15)=120$

$50 = 300 + C$

$-250 = C$

$T = 300 - 250e^{kt}$

$120 = 300 - 250e^{15k}$

$-180 = -250e^{15k}$

$\frac{18}{25} = e^{15k}$

$\frac{1}{15}\ln\frac{18}{25} = k$

5 $\quad T = 300 - 250e^{\left(\frac{1}{15}\ln\frac{18}{25}\right)t}$

$\underline{t=30}$ $\quad T = 300 - 250e^{2\ln\left(\frac{18}{25}\right)}$

$T = 170.4^\circ F$

$\underline{T=200}$

3 $\quad 200 = 300 - 250e^{\left(\frac{1}{15}\ln\frac{18}{25}\right)t}$

$-100 = -250e^{\left(\frac{1}{15}\ln\frac{18}{25}\right)t}$

$\frac{2}{5} = e^{\left(\frac{1}{15}\ln\frac{18}{25}\right)t}$

$\ln\frac{2}{5} = \left(\frac{1}{15}\ln\frac{18}{25}\right)t$

$\dfrac{15\ln\frac{2}{5}}{\ln\frac{18}{25}} = t = 41.8$ minutes.

$T = 350 + Ce^{kt}$ $\quad T(0) = 60$

$T(20) = 150$

$60 = 350 + C$

$-290 = C$

$T = 350 - 290e^{kt}$

$150 = 350 - 290e^{20k}$

$-200 = -290e^{20k}$

$\frac{20}{29} = e^{20k}$

$\frac{1}{20}\ln\frac{20}{29} = k$

$T = 350 - 290e^{\left(\frac{1}{20}\ln\frac{20}{29}\right)t}$

$\underline{t=30}$

$T = 350 - 290e^{\frac{3}{2}\ln\frac{20}{29}}$

$T = 183.9^\circ F$

$\underline{T=250}$

$250 = 350 - 290e^{\left(\frac{1}{20}\ln\frac{20}{29}\right)t}$

$-100 = -290e^{\left(\frac{1}{20}\ln\frac{20}{29}\right)t}$

$\frac{10}{29} = e^{\left(\frac{1}{20}\ln\frac{20}{29}\right)t}$

$\ln\frac{10}{29} = \left(\frac{1}{20}\ln\frac{20}{29}\right)t$

$\dfrac{20\ln\frac{10}{29}}{\ln\frac{20}{29}} = t = 57.3$ minutes.