MATH 290 – EXAM #1
Summer Session 2022

Name: ______________________

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 80 points on this exam. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may use one sheet of notes for this exam. You may not use the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may not use a calculator on this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Please write your name on all pages of this exam and your scratch paper. Good luck!

1. Given the equation $y' = \sqrt{y^2 - 4}$. Use the existence of a unique solution theorem to determine if the differential equation has a unique solution at the following given initial conditions.

$f(x,y) = \sqrt{y^2-4}$ $\quad \frac{\partial f}{\partial y} = \frac{1}{2}\frac{2y}{(y^2-4)^{1/2}} = \frac{y}{(y^2-4)^{1/2}}$

(2 points) *a.* $y(3) = 5$

unique

(2 points) *b.* $y(4) = 2$

no information

(2 points) *c.* $y(0) = 1$ no information

(3 points) *2.* Verify that $y(x) = c_1e^{5x} + c_2e^{-x}$ is a solution to the differential equation $y'' - 5y' - 6y = 0$

$y' = 5c_1e^{5x} - c_2e^{-x}$

$y'' = 25c_1e^{5x} + c_2e^{-x}$

$25c_1e^{5x} + c_2e^{-x} - 25c_1e^{5x} + 5c_2e^{-x} - 6c_1e^{5x} - 6c_2e^{-x} \neq 0$

Not a solution

3. Solve the following differential equations.

(8 points) a. $\dfrac{dy}{dx} + y = xy^4$

$y^{-4}\dfrac{dy}{dx} + y^{-3} = x$

$u = y^{-3}$

$\dfrac{du}{dx} = -3y^{-4}$

$-\dfrac{1}{3}\dfrac{du}{dx} = y^{-4}$

$-\dfrac{1}{3}\dfrac{du}{dx} + u = x$

$\dfrac{du}{dx} - 3u = -3x$

$\mu(x) = e^{-\int 3\,dx} = e^{-3x}$

$x \quad e^{-3x}$

$1 \quad \frac{1}{-3}e^{-3x}$

$0 \quad \frac{1}{9}e^{-3x}$

$e^{-3x}u = -3\int xe^{-3x}dx$

$e^{-3x}u = -3\left(\frac{1}{-3}xe^{-3x} - \frac{1}{9}e^{-3x}\right)$

$e^{-3x}y^{-3} = xe^{-3x} + \frac{1}{3}e^{-3x} + C$

(8 points) b. $(x+3y)dx - (3x+y)dy = 0$

$u = \dfrac{y}{x} \qquad y = ux$

$dy = u\,dx + x\,du$

$(x+3ux)\,dx - (3x+ux)(u\,dx + x\,du) = 0$

$x\,dx + \cancel{3ux\,dx} - \cancel{3ux\,dx} - 3x^2du - u^2x\,dx - ux^2du = 0$

$x(1-u^2)\,dx = x^2(3+u)\,du$

$\dfrac{1}{x}dx = \dfrac{3+u}{1-u^2}du \qquad \dfrac{3+u}{(1-u)(1+u)} = \dfrac{A}{1-u} + \dfrac{B}{1+u}$

$3+u = A(1+u) + B(1-u)$

$\int \frac{1}{x}dx = \int\left(\frac{2}{1-u} + \frac{1}{1+u}\right)du \qquad u=1 \quad 4 = 2A \quad A=2$

$\ln|x| = -2\ln|1-u| + \ln|1+u| + C \qquad u=-1 \quad 2 = 2B \quad B=1$

(8 points) c. $\left(2ye^{2x}+y^2\cos x\right)dx+\left(2y\sin x+e^{2x}+4y^3\right)dy=0$

$M_y = 2e^{2x} + 2y\cos x \qquad N_x = 2y\cos x + 2e^{2x}$

$\int M dx = \int (2ye^{2x} + y^2\cos x)\,dx = ye^{2x} + y^2\sin x + g(y) = f(x,y)$

$f_y = e^{2x} + 2y\sin x + g'(y) = 2y\sin x + e^{2x} + 4y^3$

$g(y) = y^4$

$e^{2x} + 2y\sin x + y^4 = C$

(8 points) d. $\dfrac{dy}{dx}+(\tan x)y=\cos^2 x$

$\mu(x) = e^{\int \tan x\, dx} = e^{-\ln|\cos x|} = \dfrac{1}{\cos x}$

$\dfrac{1}{\cos x} y = \int \cos x\, dx$

$\dfrac{1}{\cos x} y = \sin x + C$

(8 points) *e.* $x\,dy = \sqrt{1-y^2}\,dx$

$$\frac{1}{\sqrt{1-y^2}}\,dy = \frac{1}{x}\,dx$$

$$\sin^{-1} y = \ln|x| + C$$

(8 points) *4.* Solve the following initial-value problem.

$$x\frac{dy}{dx} + y = 4x+1, \quad y(1) = 8$$

$$\frac{dy}{dx} + \frac{1}{x}y = 4 + \frac{1}{x}$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$$

$$xy = \int (4x+1)\,dx$$

$$xy = 2x^2 + x + C$$

$$8 = 2 + 1 + C$$

$$C = 5$$

$$xy = 2x^2 + x + 5$$

5. Given the following differential equation: $y' = 5 + 4y - y^2$

(2 points) *a.* Determine all equilibrium solutions.

$-(y^2 - 4y - 5)$

$-(y-5)(y+1)$

$y = 5$

$y = -1$

(3 points) *b.* Determine the regions in the *xy*-plane where the solutions are increasing, and determine the regions where they are decreasing.

$-$ $+$ $-$

-1 5

increasing on $(-1, 5)$

decreasing on $(-\infty, -1) \cup (5, \infty)$

(3 points) *c.* Determine the regions in the *xy*-plane where the solutions curves are concave up, and determine the regions where they are concave down.

$y'' = (4 - 2y)\, y'$

$= 2(2-y)(5-y)(1+y)$

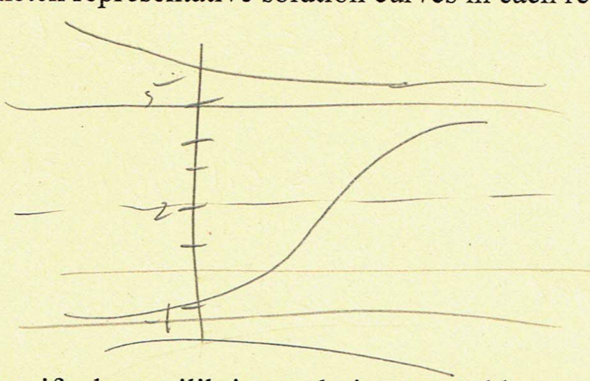
$-$ $+$ $-$ $+$

-1 2 5

Concave up $(-1, 2) \cup (5, \infty)$

Concave down $(-\infty, -1) \cup (2, 5)$

(3 points) *d.* Sketch representative solution curves in each region determined by parts *a, b* and *c*.



(2 points) *e.* Classify the equilibrium solutions as stable, unstable, or semi-stable.

$y = -1$ unstable

$y = 5$ stable

(5 points) 6. Determine the integrating factor necessary to change the following differential equations into an exact equation. Use the integrating factor to rewrite the differential equations and show that the resulting differential equation is exact. Do not solve the resulting equation.

$$\left(y^2+xy^3\right)dx+\left(5y^2-xy+y^3\sin y\right)dy=0$$

$M_y = 2y + 3xy^2 \qquad N_x = -y$

$\frac{M_y - N_x}{N} = \frac{3y+3xy^2}{y^2+xy^3} = \frac{3y(1+xy)}{y^2(1+xy)} = \frac{3}{y}$

$\frac{N_x - M_y}{M} = -\frac{3}{y} \qquad e^{-\int \frac{3}{y}dy} = e^{-3\ln|y|} = y^{-3}$

$\left(\frac{1}{y}+x\right)dx + \left(\frac{5}{y} - \frac{x}{y^2} + \sin y\right)dy = 0$

$M_y = -\frac{1}{y^2} \qquad N_x = -\frac{1}{y^2} \qquad \therefore$ exact.

(5 points) 7. Use Euler's method with the following differential equation and given step size to find y_1, y_2, y_3. Use at least four decimal places for your answers.

$h = 0.1$

$$\frac{dy}{dx} = x^2y+\sqrt{y}, \quad y(0)=1$$

$f(x,y) = x^2y + \sqrt{y} \qquad y(0) = 1 \qquad x_0 = 0 \qquad y_0 = 1$

$y_1 = 1 + 1(0.1) = 1.1 \qquad x_1 = 0.1 \quad y_1 = 1.1$

$y_2 = 1.1 + 0.10598 = 1.20598 \qquad x_2 = 0.2 \quad y_2 = 1.20598$

$y_3 = \quad 1.32062$