MATH 290 – EXAM #1 **Name:** KEY

Winter Session 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may not use a calculator on this exam. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You will need your Mt. SAC student identification card to submit this exam. Good luck!

1. For each of the following differential equations, state the order and whether the equation is linear or nonlinear.

(2 points) a. $(\sin x)y''' - (e^x \tan x)y'' + \frac{1}{x+2}y = \sec x$ 3rd-order, linear

(2 points) b. $y''y^3 + (x^2+1)y' = e^x$ 2nd-order, non-linear

(2 points) c. $y^{(4)} + 3xy''' + 7xy^2 = 3 + 8x^4$ 4th-order, non-linear.

2. Use the existence and uniqueness theorem to determine if the following initial-value problems has a unique solution.

(5 points) a. $y' = x\sin(x+y),\ y(0) = 1$

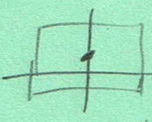
$f(x,y) = x\sin(x+y)$

$f(0,1) = 0\sin(0+1) = 0$

$\frac{\partial f}{\partial y} = x\cos(x+y)$

$\frac{\partial f}{\partial y}\Big|_{(0,1)} = 0\cos(0+1) = 0$

f and $\frac{\partial f}{\partial y}$ are defined ~~for~~ all points around (0,1)



∴ There is a unique solution for the IVP at (0,1).

(5 points) b. $y' = \sqrt{y^2-4}, \quad y(2) = 1$

$f(x,y) = \sqrt{y^2-4}$

$f(2,1) = \sqrt{2^2-4} = 0$

$\frac{\partial f}{\partial y} = \frac{1}{2}(y^2-4)^{-1/2}(2y) = \frac{y}{\sqrt{y^2-4}}$

$\frac{\partial f}{\partial y}\Big|_{(2,1)}$ is undefined.

Therefore, there is no rectangle around (2,1) where both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are defined. Therefore, the theorem gives no information.

3. Solve the following differential equations.

(10 points) a. $x\dfrac{dy}{dx}+y=x^4\ln x$

$$\frac{dy}{dx}+\frac{1}{x}y=x^3\ln x$$

$$\mu(x)=e^{\int\frac{1}{x}dx}=e^{\ln|x|}=x$$

$$\frac{d}{dx}(xy)=x^4\ln x$$

$$xy=\int x^4\ln x\,dx \rightarrow \boxed{xy=\frac{1}{5}x^5\ln x-\frac{1}{25}x^5+C}$$

$u=\ln x \quad dv=x^4dx$

$du=\frac{1}{x}dx \quad v=\frac{x^5}{5}$

$$\int x^4\ln x\,dx=\frac{1}{5}x^5\ln x-\int\frac{x^4}{5}dx$$

$$=\frac{1}{5}x^5\ln x-\frac{x^5}{25}+c$$

(10 points) b. $(ye^x+\sin y)dx+(e^x+x\cos y+5)dy=0$

$$\frac{\partial M}{\partial y}=e^x+\cos y \qquad \frac{\partial N}{\partial x}=e^x+\cos y \qquad \therefore \text{ exact}$$

$$M=ye^x+\sin y$$

$$f(x,u)=\int(ye^x+\sin y)\,dx=ye^x+x\sin y+g(y)$$

$$\frac{\partial f}{\partial y}=e^x+x\cos y+g'(y)=e^x+x\cos y+5=N$$

$$g'(y)=5$$

$$g(y)=5y$$

$$f(x,y)=ye^x+x\sin y+5y$$

solution: $\boxed{ye^x+x\sin y+5y=C}$

(10 points) *c.* $\frac{dy}{dx}+\frac{1}{2}(\tan x)y=2y^3\sin x$

$$y^{-3}\frac{dy}{dx}+\frac{1}{2}(\tan x)y^{-2}=2\sin x$$

let $u=y^{-2}$

$$\frac{du}{dx}=-2y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}\frac{du}{dx}=y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}\frac{du}{dx}+\frac{1}{2}(\tan x)u=2\sin x$$

$$\frac{du}{dx}-(\tan x)u=-4\sin x$$

$$\mu(x)=e^{-\int\tan x\,dx}=e^{\ln|\cos x|}=\cos x$$

$$\frac{d}{dx}(\cos x\, u)=-4\sin x\cos x$$

$$\cos x\, u=-\int 4\sin x\cos x\,dx$$

$v=\sin x$

$dv=\cos x\,dx$

$$-4\int v\,dv=-4\frac{v^2}{2}+C=-2v^2+C$$

$$=-2\sin^2x+C$$

$$(\cos x)y^{-2}=-2\sin^2x+C$$

(10 points) *d.* $\left(x^2+y^2\right)dx+\left(x^2-xy\right)dy=0$

$f(x,y)=(x^2+y^2)$ $g(x,y)=x^2-xy$

$f(tx,ty)=t^2x^2+t^2y^2$ $g(tx,ty)=t^2x^2-tx\,ty$

$=t^2(x^2+y^2)$ $=t^2(x^2-xy)$

Homogeneous of degree two.

let $y=ux$

$dy=u\,dx+x\,du$

$$(x^2+u^2x^2)dx+(x^2-x(ux))(u\,dx+x\,du)=0$$

$$(x^2+u^2x^2)dx+(x^2-ux^2)(u\,dx+x\,du)=0$$

$$(x^2+u^2x^2)dx+x^2u\,dx+x^3du-u^2x^2dx-ux^3du=0$$

$$(x^2+u^2x^2+x^2u-u^2x^2)dx+(x^3-ux^3)du=0$$

$$x^2(1+u)dx=x^3(u-1)du$$

$$\frac{1}{x}dx=\frac{u-1}{u+1}du$$

$$\frac{1}{x}dx=\frac{u+1-1-1}{u+1}du$$

$$\frac{1}{x}dx=\left(1-\frac{2}{u+1}\right)du$$

$$\ln|x|=u-2\ln|u+1|+C$$

$$\ln|x|=\frac{y}{x}-2\ln\left|\frac{y}{x}+1\right|+C$$

(10 points) *e.* $e^x y \dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

$$e^x y \frac{dy}{dx} = e^{-y} + e^{-2x} e^{-y}$$

$$e^x y \frac{dy}{dx} = e^{-y}(1+e^{-2x})$$

$$y e^y dy = (1+e^{-2x}) e^{-x} dx$$

$$y e^y dy = (e^{-x}+e^{-3x}) dx$$

$$\int y e^y dy = \int (e^{-x}+e^{-3x}) dx$$

$$y e^y - e^y = -e^{-x} - \frac{1}{3} e^{-3x} + C$$

(10 points) *4.* Solve the following initial-value problem.

$$(x+1)\frac{dy}{dx} + y = \ln x, \quad y(1) = 10$$

$$\frac{dy}{dx} + \frac{1}{x+1} y = \frac{1}{x+1} \ln x$$

$$\mu(x) = e^{\int \frac{1}{x+1} dx} = e^{\ln|x+1|} = x+1$$

$$\frac{d}{dx}[(x+1) y] = \ln x$$

$$(x+1) y = \int \ln x \, dx$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$\int \ln x = x \ln x - \int dx = x \ln x - x + c$$

$$(x+1) y = x \ln x - x + C$$

$$(x+1) y = x \ln x - x + 21$$

$$(1+1)(10) = 1 \ln(1) - 1 + C$$

$$20 = -1 + C$$

$$C = 21$$

5. Given the following differential equation: $y' = 3 - 2y - y^2$

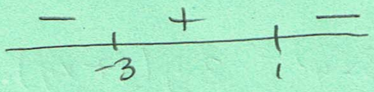
(2 points) *a.* Determine all equilibrium solutions.

$y' = -(y^2 + 2y - 3)$

$y' = -(y+3)(y-1)$

$y = -3, 1$

(4 points) *b.* Determine the regions in the *xy*-plane where the solutions are increasing, and determine the regions where they are decreasing.



decreasing on $(-\infty, -3) \cup (1, \infty)$

increasing on $(-3, 1)$

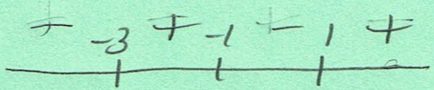
(4 points) *c.* Determine the regions in the *xy*-plane where the solutions curves are concave up, and determine the regions where they are concave down.

$y'' = (-2-2y)\, y'$

$= (-2-2y)(3-2y-y^2)$

$= -(2+2y)(-)(y+3)(y-1)$

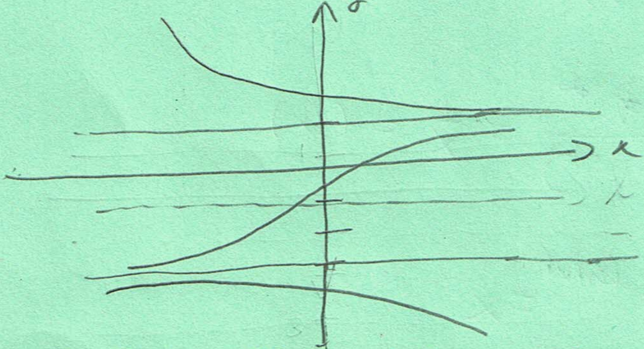
$= (2+2y)(y+3)(y-1)$



concave up on $(-3, -1) \cup (1, \infty)$

concave down on $(-\infty, -3) \cup (-1, 1)$

(4 points) *d.* Sketch representative solution curves in each region determined by parts *a, b* and *c.*



(2 points) *e.* Classify the equilibrium solutions as stable or unstable.

$y = -3$ unstable

$y = 1$ stable.

6. Determine the integrating factor necessary to change the following differential equations into an exact equation. Do not solve the differential equation.

(5 points) a. $\left(y^2+xy^3\right)dx+\left(5y^2-xy+y^3\sin y\right)dy=0$

$$M_y = 2y + 3xy^2 \qquad N_x = -y$$

$$M_y - N_x = 3y + 3xy^2$$

$$\frac{M_y - N_x}{M} = \frac{3y+3xy^2}{y^2+xy^3} = \frac{3y(1+xy)}{y^2(1+xy)} = \frac{3}{y} = f(y)$$

$$\mu(y) = e^{-\int \frac{3}{y}dy} = e^{-3\ln|y|} = y^{-3}$$

(5 points) b. $\left(3xy-2y^{-1}\right)dx+x\left(x+y^{-2}\right)dy=0$

$$M_y = 3x + 2y^{-2} \qquad N_x = 2x + y^{-2}$$

$$M_y - N_x = x + y^{-2}$$

$$\frac{M_y - N_x}{N} = \frac{x+y^{-2}}{x(x+y^{-2})} = \frac{1}{x} = f(x)$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$$