**MATH 290 – EXAM #1**
**Summer Session 2019**

**Name:** KEY

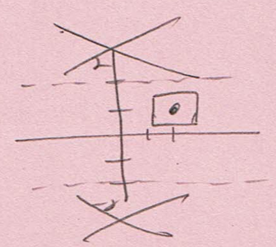
**Directions:** ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. You will need your Mt. SAC student identification card to submit this exam. Good luck!

*1.* Use the existence and uniqueness theorem to determine if each of the following initial-value problems has a unique solution, if possible.

*(4 points)* a. $y' = \sqrt{4-y^2}, \quad y(2)=1$

$f(x,y) = \sqrt{4-y^2} \qquad -2 \le y \le 2$

$\frac{\partial f}{\partial y} = \frac{-2y}{2(4-y^2)^{1/2}} \qquad -2 < y < 2$

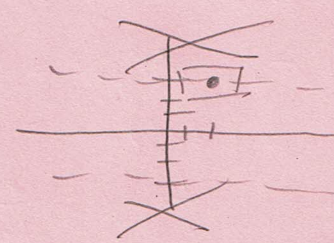


Since a rectangle can be drawn around (2,1) in which both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are both defined and continuous, there is exists a unique solution

*(4 points)* b. $y' = \sqrt{9-y^2}, \quad y(2)=3$

$f(x,y) = \sqrt{9-y^2} \qquad -3 \le y \le 3$

$\frac{\partial f}{\partial y} = \frac{-2y}{2(9-y^2)^{1/2}} \qquad -3 < y < 3$



Since it is not possible to draw a rectangle around (2,3) in which both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are both defined and continuous, the uniqueness and existence theorem gives no information

*(5 points)* *2.* Verify that $y(x)=c_1e^x+c_2e^{2x}$ is a solution to the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$$

$y'=c_1e^x+2c_2e^{2x}, \quad y''=c_1e^x+4c_2e^{2x}$

$(c_1e^x+4c_2e^{2x})-3(c_1e^x+2c_2e^{2x})+2(c_1e^x+c_2e^{2x})$

$=c_1e^x+4c_2e^{2x}-3c_1e^x-6c_2e^{2x}+2c_1e^x+2c_2e^{2x}$

$=0$ ✓

*3.* Solve the following differential equations.

*(10 points)* *a.* $(x^2+2y^2)dx-(xy)dy=0$

$M(tx,ty)=t^2x^2+2t^2y^2=t^2(x^2+2y^2)=t^2M(x,y)$

$N(tx,ty)=-(txty)=-t^2xy=t^2N(x,y)$

$\therefore$ Homogeneous of degree 2

let $u=\frac{y}{x}$, $y=ux$, $dy=x\,du+u\,dx$

$(x^2+2u^2x^2)dx-(xux)(x\,du+u\,dx)=0$

$x^2dx+2u^2x^2dx-ux^3du-u^2x^2dx=0$

$(x^2+u^2x^2)dx=ux^3du$

$x^2(1+u^2)dx=ux^3du$

$\frac{1}{x}dx=\frac{u}{1+u^2}du$

$\ln|x|=\frac{1}{2}\ln|1+u^2|+c$

$\ln|x|=\frac{1}{2}\ln|1+\frac{y^2}{x^2}|+c$

*(10 points)* b. $\dfrac{dy}{dx} + y\cot x = y^3 \sin^3 x$

Bernoulli

$u = y^{-2}$

$y^{-3}\dfrac{dy}{dx} + y^{-2}\cot x = \sin^3 x$

$\dfrac{du}{dx} = -2y^{-3}\dfrac{dy}{dx} \qquad -\dfrac{1}{2}\dfrac{du}{dx} = y^{-3}\dfrac{dy}{dx}$

$-\dfrac{1}{2}\dfrac{du}{dx} + u\cot x = \sin^3 x$

$\dfrac{du}{dx} - 2u\cot x = -2\sin^3 x$

$\mu(x) = e^{-2\int \cot x} = e^{-2\ln|\sin x|} = \dfrac{1}{\sin^2 x}$

$\dfrac{u}{\sin^2 x} = -2\int \sin x\, dx$

$\dfrac{u}{\sin^2 x} = 2\cos x + C$

$\dfrac{y^{-2}}{\sin^2 x} = 2\cos x + C$

$\dfrac{1}{y^2 \sin^2 x} = 2\cos x + C$

*(10 points)* c. $\left[\sin(xy) + xy\cos(xy) + 2x\right]dx + \left[x^2\cos(xy) + 2y\right]dy = 0$

$M_y = x\cos(xy) + x\cos(xy) - x^2 y\sin(xy)$
$= 2x\cos(xy) - x^2 y\sin(xy)$

$N_x = 2x\cos(xy) - x^2 y\sin(xy)$

$M_y = N_x \therefore$ exact.

$\int N\,dy = \int (x^2\cos(xy) + 2y)\,dy = x\sin(xy) + y^2 + g(x) = f(x,y)$

$\dfrac{\partial f}{\partial x} = \sin(xy) + xy\cos(xy) + g'(x) = M$

$\sin(xy) + xy\cos(xy) + g'(x) = \sin(xy) + xy\cos(xy) + 2x$

$g'(x) = 2x$

$g(x) = x^2 + C_1$

$f(x,y) = x\sin(xy) + y^2 + x^2 + C_1$

solution: $x\sin(xy) + y^2 + x^2 = C.$

*(10 points) d.* $x\frac{dy}{dx}+y=x^4\ln x$

$$\frac{dy}{dx}+\frac{1}{x}y=x^3\ln x$$

$$\mu(x)=e^{\int \frac{1}{x}dx}=e^{\ln|x|}=x$$

$$xy=\int x^4\ln x\,dx$$

$$u=\ln x \quad dv=x^4dx$$

$$du=\frac{1}{x}dx \quad v=\frac{x^5}{5}$$

$$\int x^4\ln x\,dx=\frac{1}{5}x^5\ln x-\frac{1}{5}\int x^4dx$$

$$=\frac{1}{5}x^5\ln x-\frac{1}{25}x^5+C$$

$$xy=\frac{1}{5}x^5\ln x-\frac{1}{25}x^5+C$$

*(10 points) e.* $x\left(1+y^2\right)^{1/2}dx-y\left(1+x^2\right)^{1/2}dy=0$

$$x(1+y^2)^{1/2}dx=y(1+x^2)^{1/2}dy$$

$$\frac{x}{(1+x^2)^{1/2}}dx=\frac{y}{(1+y^2)^{1/2}}dy$$

$$\sqrt{1+x^2}=\sqrt{1+y^2}+C$$

*(10 points)* 4. Solve the following initial-value problem.

$$y' - (\sin x)\, y = 2\sin x, \quad y(\pi/2) = 1$$

$\mu(x) = e^{-\int \sin x\, dx} = e^{\cos x}$

$y e^{\cos x} = 2\int \sin x\, e^{\cos x}\, dx$

$y e^{\cos x} = -2e^{\cos x} + C$

$y(\pi/2) = 1 \qquad 1 \cdot e^{\cos \pi/2} = -2e^{\cos \pi/2} + C$

$1 = -2 + C$

$C = 3$

$y e^{\cos x} = -2e^{\cos x} + 3$

*(6 points)* 5. Use Euler's method with the following function and given step size to find $y_1$, $y_2$, and $y_3$. Use at least four decimal places for your answers.

$$y' = x^2 y + y^2, \quad y(1) = 2, \quad h = 0.1$$

$f(2,1) = (1)^2(2) + (2)^2 = 6$

$y_1 = 2 + 6 \cdot (0.1) = 2.6$

$x_1 = 1.1$

$y_1 = 2.6$

$f(1.1, 2.6) = (1.1)^2(2.6) + (2.6)^2 = 9.906$

$y_2 = 2.6 + 9.906(0.1) = 3.5906$

$x_2 = 1.2$

$y_2 = 3.5906$

$f(1.2, 3.5906) = (1.2)^2(3.5906) + (3.5906)^2 = 18.06287236$

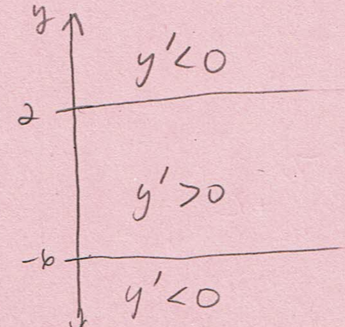
$y_3 = 3.5906 + 18.06287236(0.1) = 5.3969$

6. Given the following differential equation: $y' = 12 - 4y - y^2$

*(2 points)* *a.* Determine all equilibrium solutions.

$$-(y^2+4y-12)$$
$$-(y+6)(y-2)=0 \qquad y=-6,\ y=2$$

*(3 points)* *b.* Determine the intervals along the *y*-axis on the phase portrait where the solutions are increasing, and determine the regions where they are decreasing.

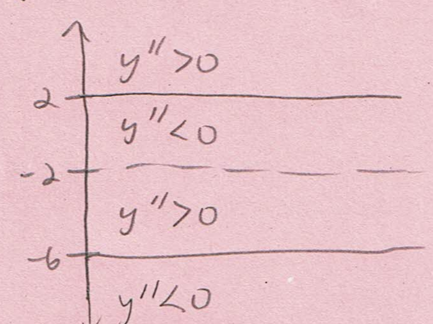



increasing on $(-6,2)$

decreasing on $(-\infty,-6)\cup(2,\infty)$

*(3 points)* *c.* Determine the intervals along the *y*-axis on the phase portrait where the solutions curves are concave up and the intervals where they are concave down.

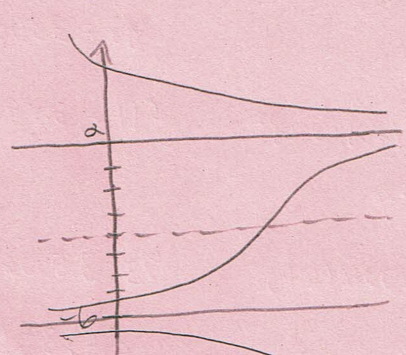
$$y'' = (-4-2y)y' = -2(2+y)(-(y+6)(y-2)) = 2(2+y)(y+6)(y-2)$$




Concave up $(-6,-2)\cup(2,\infty)$

Concave down $(-\infty,-6)\cup(-2,2)$

*(3 points)* *d.* Sketch representative solution curves in each region determined by parts *a, b* and *c*.




*(2 points)* *e.* Classify each equilibrium solution as stable, unstable, or semi-stable.

$y=2$ stable

$y=-6$ unstable.

7. Determine the integrating factor necessary to change the following differential equations into an exact equation. Then, verify that the integration factor makes the resulting differential equation exact.

*(5 points) a.* $\left(3xy-2y^{-1}\right)dx+x\left(x+y^{-2}\right)dy=0$

$M_y = 3x+2y^{-2} \quad N_x = 2x+y^{-2}$

$M_y - N_x = (3x+2y^{-2})-(2x+y^{-2}) = x+y^{-2}$

$\dfrac{M_y-N_x}{N} = \dfrac{x+y^{-2}}{x(x+y^{-2})} = \dfrac{1}{x} \qquad \mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$

$(3x^2y-2xy^{-1})dx+(x^3+x^2y^{-2})dy=0$

$M_y = 3x^2+2xy^{-2} \quad N_y = 3x^2+2xy^{-2} \quad \therefore$ exact

*(5 points) b.* $(xy)dx+\left(2x^2+3y^2-20\right)dy=0$

$M_y = x \qquad N_x = 4x$

$\dfrac{N_x-M_y}{M} = \dfrac{4x-x}{xy} = \dfrac{3x}{xy} = \dfrac{3}{y} \qquad \mu(y) = e^{\int \frac{3}{y}dy} = e^{3\ln|y|} = y^3$

$(xy^4)dx+(2x^2y^3+3y^5-20y^3)dy=0$

$M_y = 4xy^3 \qquad N_x = 4xy^3 \quad \therefore$ exact