MATH 290 – EXAM #2
Spring Semester 2019

Name: KEY

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(6 points) 1. Use Euler's method with the following function and given step size to find y_1, y_2, and y_3. Use at least four decimal places for your answers.

$$y' = xy - y^2, \quad y(2) = 1, \quad h = 0.1$$

$x_0 = 2 \quad y_0 = 1 \qquad y_1 = 1 + 0.1[2(1) - (1)^2] = 1.1$

$x_1 = 2.1 \quad y_1 = 1.1 \qquad y_2 = 1.1 + 0.1[2.1(1.1) - (1.1)^2] = 1.21$

$x_2 = 2.2 \quad y_2 = 1.21 \qquad y_3 = 1.21 + 0.1[2.2(1.21) - (1.21)^2] = 1.32979$

(9 points) 2. Consider a 100-volt electromotive force that is applied to an *RC*-series circuit in which the resistance is 5 ohms and the capacitance is 1/50 Farads. If the capacitor is uncharged initially, determine the current in the circuit.

$R\frac{dq}{dt} + \frac{1}{C}q = E$

$5\frac{dq}{dt} + 50q = 100$

$\frac{dq}{dt} + 10q = 20$

$\mu(t) = e^{\int 10\,dt} = e^{10t}$

$\frac{d}{dt}\left[e^{10t}q\right] = 20e^{10t}$

$e^{10t}q = 2e^{10t} + C$

$q = 2 + Ce^{-10t}$

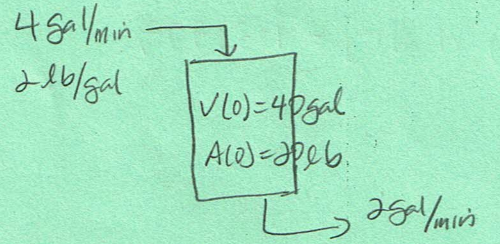
$q(0)=0 \quad 0 = 2 + C \Rightarrow C = -2$

$q(t) = 2 - 2e^{-10t}$

$i(t) = \frac{dq}{dt}$

$i(t) = 20e^{-10t}$

(9 points) 3. A tank initially contains 40 gallons of a solution in which 20 lb of salt are dissolved. A solution containing 2 lb/gal of salt runs into the tank at the rate of 4 gal/min. The well-stirred mixture flows out of the tank at the rate of 2 gal/min. If the tank has a volume capacity of 80 gallons, find the concentration of the salt in the tank just as the tank overflows. Round your answer to the nearest tenth.



$\frac{dV}{dt} = 4 - 2$

$dV = 2\,dt$

$V = 2t + C_1$

$t=0: V=40$ $\quad 40 = C_1$

$V = 2t + 40$

$\frac{dA}{dt} = C_{in}r_{in} - C_{out}r_{out}$

$\frac{dA}{dt} = 2\frac{lb}{gal}\cdot 4\frac{gal}{min} - \frac{A}{V}\frac{lb}{gal}\cdot 2\frac{gal}{min}$

$\frac{dA}{dt} + \frac{2}{2t+40}A = 8$

$\frac{dA}{dt} + \frac{1}{t+20}A = 8$

$\mu(t) = e^{\int \frac{1}{t+20}dt} = e^{\ln|t+20|}$

$= t+20$

$\frac{d}{dt}\left[(t+20)A\right] = 8(t+20)$

$(t+20)A = 4(t+20)^2 + C_2$

$t=0, A=20:$

$(20)(20) = 4(20)^2 + C_2$

$400 = 4(400) + C_2$

$C_2 = -1200$

$(t+20)A = 4(t+20)^2 - 1200$

$V = 80$ gal: $\quad 80 = 2t + 40$

$40 = 2t$

$t = 20$ min

$(40)A = 4(40)^2 - 1200$

$A = 130$ lbs

concentration: $130\,lb / 80\,gal \doteq 1.625\,lb/gal$

4. A metal bar whose temperature is 350°F is placed in a room whose temperature is a constant 70°F. After two minutes, the temperature of the bar is 210°F.

(7 points) *a.* Find the temperature of the substance after four minutes.

$T = T_m + Ce^{kt}$

$T = 70 + Ce^{kt}$

$T(0) = 350, \quad T(2) = 210$

$t=0$: $350 = 70 + C$

$280 = C$

$T = 70 + 280e^{kt}$

$t=2$: $210 = 70 + 280e^{2k}$

$140 = 280e^{2k}$

$\frac{1}{2} = e^{2k}$

$\ln \frac{1}{2} = 2k \quad k = \frac{1}{2}\ln \frac{1}{2}$

$T = 70 + 280\,e^{\frac{1}{2}\ln(\frac{1}{2})t}$

$t=4$: $T = 70 + 280\,e^{(\frac{1}{2}\ln\frac{1}{2})4}$

$T = 140°F$

(2 points) *b.* Find the time required for the bar to cool to 100°F

$100 = 70 + 280\,e^{(\frac{1}{2}\ln\frac{1}{2})t}$

$30 = 280\,e^{(\frac{1}{2}\ln\frac{1}{2})t}$

$\frac{3}{28} = e^{(\frac{1}{2}\ln\frac{1}{2})t}$

$\ln\frac{3}{28} = (\frac{1}{2}\ln\frac{1}{2})t$

$t = \dfrac{\ln\frac{3}{28}}{\frac{1}{2}\ln\frac{1}{2}}$

$t \doteq 6.4$ minutes.

(4 points) 5. Given that $y_1(x) = e^{6x}$ and $y_2(x) = e^{2x}$ are solutions to the differential equation $y'' - 8y' + 12y = 0$. Verify that y_1 and y_2 form a fundamental set of solutions to the differential equation on the interval $(-\infty, \infty)$.

$$\begin{vmatrix} e^{6x} & e^{2x} \\ 6e^{6x} & 2e^{2x} \end{vmatrix} = 2e^{8x} - 6e^{8x} = -4e^{8x} \neq 0$$

$\therefore$ y_1 and y_2 are linearly independent.

$\therefore$ y_1 and y_2 form a fundamental set of solutions.

6. Two chemicals A and B are combined to form a chemical C. The rate of reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 10 grams of A and 30 grams of B, and for every 2 grams of A, 3 grams of B are used. It is observed that 10 grams of C are formed in 5 minutes.

(9 points) *a.* How much of C is formed in 30 minutes?

$$\frac{dx}{dt} \propto \left(10 - \frac{2}{5}x\right)\left(30 - \frac{3}{5}x\right)$$

$$\frac{dx}{dt} = k(25-x)(50-x)$$

$$\frac{dx}{(25-x)(50-x)} = k\,dt$$

$$\int \frac{dx}{(25-x)(50-x)} = \int k\,dt$$

$$\frac{1}{(25-x)(50-x)} = \frac{A}{25-x} + \frac{B}{50-x}$$

$$1 = A(50-x) + B(25-x)$$

$x = 25 \quad A = \frac{1}{25}$

$x = 50 \quad B = -\frac{1}{25}$

$$\int\left(\frac{1/25}{25-x} + \frac{-1/25}{50-x}\right) dx = \int k\,dt$$

$$-\frac{1}{25}\ln|25-x| + \frac{1}{25}\ln|50-x| = kt + C_1$$

$$\frac{1}{25}\ln\left|\frac{50-x}{25-x}\right| = kt + C_1$$

$$\ln\left|\frac{50-x}{25-x}\right|^{1/25} = kt + C_1$$

$$\left(\frac{50-x}{25-x}\right)^{1/25} = Ce^{kt}$$

$x=0,\ t=0: \quad 2^{1/25} = C$

$$\left(\frac{50-x}{25-x}\right)^{1/25} = 2^{1/25}e^{kt}$$

$$\frac{50-x}{25-x} = 2e^{25kt}$$

$x = 10 \quad t = 5:$

$$\frac{40}{15} = 2e^{125k}$$

$$\frac{4}{3} = e^{125k}$$

$$\frac{1}{125}\ln 4/3 = k$$

$t = 30$

$$\frac{50-x}{25-x} = 2e^{25\left(\frac{1}{125}\ln\frac{4}{3}\right)30}$$

$$\frac{50-x}{25-x} = 2e^{6\ln\frac{4}{3}}$$

$$50 - x = (25-x)\,2e^{6\ln 4/3}$$

$$\left(2e^{6\ln 4/3} - 1\right)x = 50e^{6\ln 4/3} - 50$$

$$x = \frac{50e^{6\ln 4/3} - 50}{2e^{6\ln 4/3} - 1}$$

$$x = \frac{50\left(\frac{4}{3}\right)^6 - 50}{2\left(\frac{4}{3}\right)^6 - 1}$$

$x \doteq 22.6$ g

(2 points) *b.* What is the limiting amount of C?

$x = 25$ grams.

(5 points) 7. Determine if the following set of functions is linearly independent or linearly dependent.

$$f_1(x) = 2 + x^2,\ f_2(x) = x - 3x^2,\ f_3(x) = 2x^2$$

$$\begin{vmatrix} 2+x^2 & x-3x^2 & 2x^2 \\ 2x & 1-6x & 4x \\ 2 & -6 & 4 \end{vmatrix} = (2+x^2)[4(1-6x)+24x] - (x-3x^2)[8x-8x] + 2x^2[-12x-2(1-6x)]$$

$$= (2+x^2)[4-24x+24x] + 2x^2[-12x-2+12x]$$

$$= (2+x^2)(4) + 2x^2(-2) = 8 \neq 0$$

∴ The functions are linearly independent.

(8 points) 8. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not inflected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

$$\frac{dx}{dt} \propto x(1000-x)$$

$$\frac{dx}{dt} = kx(1000-x)$$

$$\frac{dx}{dt} = x(1000k - kx)$$

$a = 1000k \quad b = k$

$P_0 = 1$

$$x(t) = \frac{aP_0}{(a-bP_0)e^{-at} + bP_0}$$

$$x(t) = \frac{1000k}{(1000k-k)e^{-1000kt} + k}$$

$$x(t) = \frac{1000}{999e^{-1000kt} + 1}$$

$$50 = \frac{1000}{999e^{-4000k} + 1}$$

$$50(999e^{-4000k}) + 50 = 1000$$

$$50(999e^{-4000k}) = 950$$

$$999e^{-4000k} = 19$$

$$e^{-4000k} = \frac{19}{999}$$

$$-4000k = \ln\frac{19}{999}$$

$$k = -\frac{1}{4000}\ln\frac{19}{999}$$

$$x(6) = \frac{1000}{999e^{\left(\frac{1}{4}\ln\frac{19}{999}\right)(6)} + 1}$$

$x(6) \doteq 276$ students.

9. Solve the following differential equations.

(4 points) a. $y'' - 4y' + 13y = 0$

$m^2 - 4m + 13 = 0$

$m = \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2(1)}$

$= \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$

$= \frac{4 \pm 6i}{2} = 2 \pm 3i$

$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

(4 points) b. $2y'' + 5y' - 12y = 0$

$2m^2 + 5m - 12 = 0$

$(2m - 3)(m + 4) = 0$

$m = 3/2, -4$

$y = c_1 e^{3/2 x} + c_2 e^{-4x}$

(4 points) c. $4y'' - 20y' + 25y = 0$

$4m^2 - 20m + 25 = 0$

$(2m - 5)(2m - 5) = 0$

$y = c_1 e^{5/2 x} + c_2 x e^{5/2 x}$

(4 points) d. $y''' - 9y'' - 4y' + 36y = 0$

$m^3 - 9m^2 - 4m + 36 = 0$

$m^2(m - 9) - 4(m - 9) = 0$

$(m - 9)(m^2 - 4) = 0$

$(m - 9)(m + 2)(m - 2) = 0$

$y = c_1 e^{9x} + c_2 e^{-2x} + c_3 e^{2x}$

(8 points) *10.* Given that $y_1(x) = \sin 4x$ is a solution to $y'' + 16y = 0$. Use reduction of order to find a second solution, $y_2(x)$.

$$y_2 = y_1(x) \int \frac{e^{\int p(x)\,dx}}{y_1^2(x)}\,dx$$

$$= \sin 4x \int \frac{e^{\int 0\,dx}}{\sin^2 4x}\,dx$$

$$= \sin 4x \int \frac{e^{C_1}}{\sin^2 4x}\,dx$$

$$= C_2 \sin 4x \int \csc^2 4x\,dx$$

$$= C_2 \sin 4x \left(\frac{-\cot 4x}{4}\right) +$$

$$= C_3 \cos 4x$$

$$y_2 = \cos 4x$$

(3 points) *11.* Given the following general solution to the given differential equation.

$$y = c_1 e^{5x} + c_2 e^{x} + x^2 + 3x; \quad y'' - 6y' + 5y = 5x^2 + 3x - 16$$

Verify that y is a solution to the differential equation.

$$y' = 5c_1 e^{5x} + c_2 e^{x} + 2x + 3$$

$$y'' = 25c_1 e^{5x} + c_2 e^{x} + 2$$

$$(25c_1 e^{5x} + c_2 e^{x} + 2) - 6(5c_1 e^{5x} + c_2 e^{x} + 2x + 3) + 5(c_1 e^{5x} + c_2 e^{x} + x^2 + 3x)$$

$$= 25c_1 e^{5x} + c_2 e^{x} + 2 - 30c_1 e^{5x} - 6c_2 e^{x} - 12x - 18 + 5c_1 e^{5x} + 5c_2 e^{x} + 5x^2 + 15x$$

$$= 5x^2 + 3x - 16$$

(8 points) *12.* Solve the following differential equation by using the superposition method.

$$y'' - 6y' + 8y = 2e^{3x}$$

$m^2 - 6m + 8 = 0$

$(m-4)(m-2) = 0$

$m = 4, 2$

$y_c = c_1 e^{4x} + c_2 e^{2x}$

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$9Ae^{3x} - 6(3Ae^{3x}) + 8(Ae^{3x}) = 2e^{3x}$

$-Ae^{3x} = 2e^{3x}$

$A = -2$

$y_p = -2e^{3x}$

$y = c_1 e^{4x} + c_2 e^{2x} - 2e^{3x}$

13. Given the following general solution to the given differential equation.

$$y = c_1x^2 + c_2x^4 + 3; \qquad x^2y'' - 5xy' + 8y = 24$$

Determine if the following boundary conditions yield no solution, one solution, or infinitely many solutions.

(2 points) *a.* $y(0) = 3,\ y(1) = 0$

$3 = 3$

$c_1 + c_2 + 3 = 0$

$c_1 + c_2 = -3$ c_2 is free $\therefore$ infinitely many solutions.

(2 points) *b.* $y(-1) = 0,\ y(1) = 5$

$0 = c_1 + c_2 + 3$ $c_1 + c_2 = -3$

$5 = c_1 + c_2 + 3$ $c_1 + c_2 = 2$ > not possible

$\therefore$ no solution

(2 points) *c.* $y(1) = 3,\ y(2) = 15$

$3 = c_1 + c_2 + 3$

$15 = 4c_1 + 16c_2 + 3$

$c_1 + c_2 = 0$

$4c_1 + 16c_2 = 12$ $\therefore$ one solution