MATH 290 – EXAM #2 **Name:** 1&1

Summer Session 2019

Directions: ***<u>Please show all work for maximum credit.</u>*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(9 points) 1. Consider a 20-volt electromotive force that is applied to an *RC*-series circuit in which the resistance is 0.5 ohm and the capacitance is 0.1 farad. Given that the capacitor has zero initial charge, determine the current in the circuit after 0.25 seconds.

$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

$\frac{1}{2}\frac{dq}{dt} + \frac{1}{1/10}q = 20$

$\frac{1}{2}\frac{dq}{dt} + 10q = 20$

$\frac{dq}{dt} + 20q = 40$

$\mu(t) = e^{\int 20dt} = e^{20t}$

$\frac{d}{dt}[e^{20t}q] = 40e^{20t}$

$e^{20t}q = \int 40e^{20t}dt$

$e^{20t}q = 2e^{20t} + C$

$q = 2 + Ce^{-20t}$

$q(0)=0 \qquad 0 = 2 + Ce^{-20(0)}$

$C = -2$

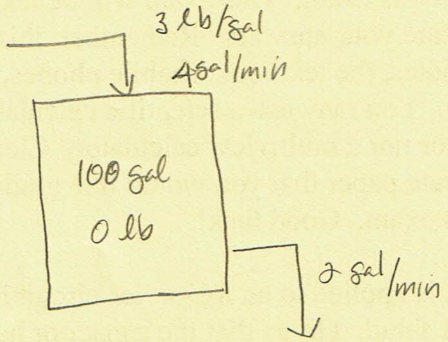
$q = 2 - 2e^{-20t}$

$i = 40e^{-20t}$

$i(0.25) = 40e^{-20(0.25)}$

$= 0.2695$ amps.

(9 points) 2. A tank initially contains 100 gallons of pure water. A solution containing 3 lb/gal of salt runs into the tank at the rate of 4 gal/min. The well-stirred mixture flows out of the tank at the rate of 2 gal/min. Find the concentration of the salt in the tank 30 minutes after the process starts. Round your answer to the nearest tenth.



$V(0) = 100$ gal

$A(0) = 0$ lb.

$$\frac{dV}{dt} = 4 - 2 = 2$$

$$dV = 2\,dt$$

$$V = 2t + C_1$$

$$100 = 2(0) + C_1$$

$$C_1 = 100$$

$$V = 2t + 100$$

$$\frac{dA}{dt} = 3\frac{lb}{gal} \cdot 4\frac{gal}{min} - \frac{A}{V}\frac{lb}{gal} \cdot 2\frac{gal}{min}$$

$$\frac{dA}{dt} = 12 - \frac{A}{2t+100} \cdot 2$$

$$\frac{dA}{dt} + \frac{1}{t+50} = 12$$

$$\mu(t) = e^{\int \frac{1}{t+50}dt} = e^{\ln|t+50|} = t+50$$

$$\frac{d}{dt}\left[(t+50)A\right] = 12(t+50)$$

$$(t+50)A = 12\int (t+50)\,dt$$

$$(t+50)A = 12\frac{(t+50)^2}{2} + C_2$$

$$A = 6(t+50) + C_2(t+50)^{-1}$$

$$0 = 300 + \frac{C_2}{50}$$

$$-300 = \frac{C_2}{50}$$

$$-15000 = C_2$$

$$A = 6(t+50) - 15000(t+50)^{-1}$$

$t = 30$

$$A = 6(30+50) - 15000(30+50)^{-1}$$

$A = 292.5$ lb.

$V = 2(30) + 100 = 160$ gal

$$\text{Concentration} = \frac{292.5 \text{ lb}}{160 \text{ gal}}$$

$$\doteq 1.8 \frac{lb}{gal}$$

(7 points) *3.* The number of bacteria in a culture grows at a rate that is proportional to the number present. Initially, there were 10 bacteria in the culture. If the doubling time of the culture if 3 hours, find the number of bacteria that are present after 24 hours.

$P = P_0 e^{kt}$

$P = 10 e^{kt}$

$20 = 10 e^{3k}$

$2 = e^{3k}$

$\frac{\ln 2}{3} = k$

$P = 10 e^{\left(\frac{\ln 2}{3}\right)t}$

$P = 10 e^{\left(\frac{\ln 2}{3}\right)24}$

$P = 10 e^{8 \ln 2}$

$P = 2560$ bacteria

4. A hot coal was pulled out of a furnace and allowed to cool at room temperature (75°F). If, after 10 minutes, the temperature of the coal was 415°F, and after 20 minutes, its temperature was 347°F, find the following:

(8 points) *a.* The temperature of the furnace.

$T = T_m + Ce^{kt}$

$T = 75 + Ce^{kt}$

$415 = 75 + Ce^{k \cdot 10}$

$347 = 75 + Ce^{k \cdot 20}$

$340 = Ce^{10k} \rightarrow C = 340 e^{-10k}$

$272 = Ce^{20k} \rightarrow 272 = 340 e^{-10k} e^{20k}$

$272 = 340 e^{10k}$

$\frac{272}{340} = e^{10k}$

$\frac{1}{10} \ln \frac{272}{340} = k$

$C = 340 e^{-10\left(\frac{1}{10} \ln \frac{272}{340}\right)}$

$C = 340 e^{-\left(\ln \frac{272}{340}\right)}$

$C = 340\left(\frac{340}{272}\right) = 425$

$T = 75 + 425 e^{\left(\frac{1}{10} \ln \frac{272}{340}\right)t}$

$t = 0$

$T = 75 + 425 = 500°F$

(2 points) *b.* The time when the temperature of the coal is 100°F.

$100 = 75 + 425 e^{\left(\frac{1}{10} \ln \frac{272}{340}\right)t}$

$\frac{25}{425} = e^{\left(\frac{1}{10} \ln \frac{272}{340}\right)t}$

$\ln \frac{1}{17} = \frac{1}{10} \ln \frac{272}{340} t$

$10 \ln \frac{1}{17} = \left(\ln \frac{272}{340}\right)t$

$t = \frac{10 \ln \frac{1}{17}}{\ln \frac{272}{340}} \doteq 127$ minutes.

5. Two chemicals A and B are combined to form a chemical C. The rate of reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 30 grams of A and 40 grams of B, and for each gram of B, 3 grams of A are used. It is observed that 5 grams of C are formed in 5 minutes.

(9 points) *a.* How much of C is formed in 15 minutes?

$$\frac{dx}{dt} \propto \left(30-\tfrac{3}{4}x\right)\left(40-\tfrac{1}{4}x\right)$$

$$\frac{dx}{dt} = K(40-x)(160-x)$$

$$\frac{dx}{(40-x)(160-x)} = k\,dt$$

$$\frac{1}{(40-x)(160-x)} = \frac{A}{40-x} + \frac{B}{160-x}$$

$$1 = A(160-x) + B(40-x)$$

$x=40 \quad A = 1/120$

$x=160 \quad B = -1/120$

$$\left(\frac{1/120}{40-x} - \frac{1/120}{160-x}\right)dx = k\,dt$$

$$\int\left(\frac{1/120}{40-x} - \frac{1/120}{160-x}\right)dx = \int k\,dt$$

$$\frac{1}{120}\ln|160-x| - \frac{1}{120}\ln|40-x| = kt + C_1$$

$$\frac{1}{120}\ln\left|\frac{160-x}{40-x}\right| = kt + C_1$$

$$\ln\left|\frac{160-x}{40-x}\right| = 120kt + 120C_1$$

$$\frac{160-x}{40-x} = C_2 e^{120kt}$$

$t=0,\ x=0$

$$4 = C_2$$

$$\frac{160-x}{40-x} = 4e^{120kt}$$

$t=5,\ x=5$

$$\frac{155}{35} = 4e^{120k(5)}$$

$$\frac{31}{7} = 4e^{600k}$$

$$\frac{31}{28} = e^{600k}$$

$$\frac{1}{600}\ln\frac{31}{28} = k$$

$$\frac{160-x}{40-x} = 4e^{120\left(\frac{1}{600}\ln\frac{31}{28}\right)t}$$

$$\frac{160-x}{40-x} = 4e^{\frac{1}{5}\left(\ln\frac{31}{28}\right)t}$$

$$\frac{160-x}{40-x} = 4\left(\frac{31}{28}\right)^{\frac{1}{5}t}$$

$t=15 \quad \dfrac{160-x}{40-x} = 4\left(\dfrac{31}{28}\right)^3$

$$160-x = 4\left(\frac{31}{28}\right)^3(40-x)$$

$$\left(4\left(\frac{31}{28}\right)^3 - 1\right)x = 160\left(\frac{31}{28}\right)^3 - 160$$

$$x = \frac{160\left(\frac{31}{28}\right)^3 - 160}{4\left(\frac{31}{28}\right)^3 - 1}$$

$x = 12.9\text{ g}$

(3 points) *b.* What is the limiting amount of C?

$40 - x \rightarrow \quad x = 40\text{ g}$

(9 points) 6. Suppose a student carrying a flu virus returns to an isolated college campus of 2000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not inflected, determine the number of infected students after 10 days if it is further observed that after 6 days $x(6)=100$.

$$\frac{dx}{dt} \propto x(2000-x) \quad , \quad x(0)=1$$

$$\frac{dx}{dt} = Kx(2000-x)$$

$$\frac{dx}{dt} = x(2000k - kx)$$

$$x(t) = \frac{2000k \cdot 1}{k \cdot 1 + (2000k - k \cdot 1)e^{-2000kt}}$$

$$x(t) = \frac{2000k}{k + 1999ke^{-2000kt}}$$

$$x(t) = \frac{2000}{1+1999e^{-2000kt}}$$

$x(6)=100$

$$100 = \frac{2000}{1+1999e^{-2000k(6)}}$$

$$100(1+1999e^{-12000k}) = 2000$$

$$100 + 199900e^{-12000k} = 2000$$

$$199900e^{-12000k} = 1900$$

$$e^{-12000k} = \frac{19}{1999}$$

$$-12000k = \ln\frac{19}{1999}$$

$$k = -\frac{1}{12000}\ln\frac{19}{1999}$$

$$x(t) = \frac{2000}{1+1999e^{\left(\frac{1}{6}\ln\frac{19}{1999}\right)t}} = \frac{2000}{1+1999\left(\frac{19}{1999}\right)^{\frac{1}{6}t}}$$

$$x(10) = \frac{2000}{1+1999\left(\frac{19}{1999}\right)^{\frac{1}{6}(10)}}$$

$x(10) \doteq 1080$ people.

(6 points) 7. Verify that $y_1(x)=\sin 5x$ and $y_2(x)=\cos 5x$ form a fundamental set of solutions to the differential equation $y''-25y=0$ on the interval $(-\infty,\infty)$.

$y_1' = 5\cos 5x \qquad y_2' = -5\sin 5x$

$y_1'' = -25\sin 5x \qquad y_2'' = -25\cos 5x$

$-25\sin 5x - 25(\sin 5x) = -50\sin 5x \neq 0$

$-25\cos 5x - 25(\cos 5x) = -50\cos 5x \neq 0$

So, y_1 and y_2 are not solutions to the differential equation.

$\therefore$ y_1 and y_2 do not form a fundamental set of solutions to the differential equation

(6 points) 8. Determine if the following set of functions is linearly independent or linearly dependent.

$$f_1(x) = 4 + x^2,\ f_2(x) = 3 + x,\ f_3(x) = 5x + 2x^2$$

$$\begin{vmatrix} 4+x^2 & 3+x & 5x+2x^2 \\ 2x & 1 & 5+4x \\ 2 & 0 & 4 \end{vmatrix}$$

$= 2\left[(3+x)(5+4x) - (5x+2x^2)\right] + 4\left[(4+x^2)(1) - 2x(3+x)\right]$

$= 2\left[15+12x+5x+4x^2-5x-2x^2\right] + 4\left[4+x^2-6x-2x^2\right]$

$= 2\left[2x^2+12x+15\right] + 4\left[-x^2-6x+4\right]$

$= 4x^2+24x+30-4x^2-24x+16$

$= 46 \neq 0$ $\therefore f_1, f_2, f_3$ are linearly independent

(9 points) 9. Given that $y_1(x) = x^{-3}$ is a solution to $x^2y'' + 5xy' + 3y = 0$. Use reduction of order to find a second solution, $y_2(x)$.

$y'' + \frac{5}{x}y' + \frac{3}{x^2} = 0$

$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2}dx$

$= x^{-3}\int \frac{e^{-\int \frac{5}{x}dx}}{(x^{-3})^2}dx$

$= x^{-3}\int \frac{e^{-5\ln|x|}}{x^{-6}}dx$

$= x^{-3}\int \frac{x^{-5}}{x^{-6}}dx$

$= x^{-3}\int x\,dx$

$= x^{-3}\cdot\frac{x^2}{2} = \frac{1}{2}x^{-1}$

$y_2 = x^{-1}$

10. Solve the following differential equations.

(4 points) *a.* $2y''+5y'-12y=0$

$2m^2+5m-12=0$

$(2m-3)(m+4)=0$

$y=c_1e^{3/2x}+c_2e^{-4x}$

(4 points) *b.* $4y''-2y'+3y=0$

$4m^2-2m+3=0$

$m=\frac{-(-2)\pm\sqrt{(-2)^2-4(4)(3)}}{2(4)}$

$=\frac{2\pm\sqrt{4-48}}{8}$

$=\frac{2\pm\sqrt{-44}}{8}$

$y=e^{\frac{1}{4}x}\left[c_1\cos\left(\frac{\sqrt{11}}{4}x\right)+c_2\sin\left(\frac{\sqrt{11}}{4}x\right)\right]$

$=\frac{2\pm 2i\sqrt{11}}{8}$

$=\frac{1}{4}\pm\frac{1}{4}i\sqrt{11}$

(4 points) *c.* $y'''-y''-4y'+4y=0$

$m^3-m^2-4m+4=0$

$(m^3-m^2)-(4m-4)=0$

$m^2(m-1)-4(m-1)=0$

$(m-1)(m^2-4)=0$

$(m-1)(m+2)(m-2)=0$

$m=1,-2,2$

$y=c_1e^x+c_2e^{-2x}+c_3e^{2x}$

(4 points) *d.* $4y'' - 20y' + 25y = 0$

$4m^2 - 20m + 25 = 0$

$(2m-5)(2m-5) = 0$

$m = 5/2$ multiplicity 2

$y = c_1 e^{5/2x} + c_2 x e^{5/2x}$

11. Given the following general solution to the given differential equation.

$$y = c_1 x^2 + c_2 x^4 + 3; \qquad x^2 y'' - 5xy' + 8y = 24$$

Determine if the following boundary conditions yield no solution, one solution, or infinitely many solutions.

(3 points) *a.* $y(0) = 3,\ y(1) = 0$

$3 = 0 + 0 + 3$

$0 = c_1 + c_2 + 3$

infinite number of solutions.

c_1 and c_2 are free

(3 points) *b.* $y(-1) = 0,\ y(1) = 5$

$0 = c_1 + c_2 + 3$
$5 = c_1 + c_2 + 3$ $\rightarrow$ $c_1 + c_2 = -3$
$c_1 + c_2 = 2$ > not possible.

No solution

(3 points) *c.* $y(1) = 3,\ y(2) = 15$

$3 = c_1 + c_2 + 3$
$15 = 4c_1 + 16c_2 + 3$ $\rightarrow$ $-4(c_1 + c_2 = 0)$
$4c_1 + 16c_2 = 12$

$-4c_1 - 4c_2 = 0$
$4c_1 + 16c_2 = 12$

$12c_2 = 12$

$c_2 = 1 \quad c_1 = -1$

One solution