MATH 290 EXAM 8

① $Ri + \frac{1}{C} q = E(t)$

$6\frac{dq}{dt} + 5q = 60$

$\frac{dq}{dt} + \frac{5}{6}q = 10$

$\mu(t) = e^{\int \frac{5}{6} dt} = e^{\frac{5}{6}t}$

$e^{\frac{5}{6}t} q = 10 \int e^{5/6 t} dt$

6/

$e^{5/6 t} q = 12 e^{5/6 t} + C$

$0 = 12 + C$

$C = -12$

$e^{5/6 t} q = 12 e^{5/6 t} - 12$

$q(t) = 12 - 12 e^{-\frac{5}{6}t}$

$i(t) = 10 e^{-\frac{5}{6}t}$

$Ri + \frac{1}{C} q = E(t)$

$4\frac{dq}{dt} + 5q = 40$

$\frac{dq}{dt} + \frac{5}{4}q = 10$

$\mu(t) = e^{\int \frac{5}{4} dt} = e^{\frac{5}{4}t}$

$e^{\frac{5}{4}t} q = 10 \int e^{\frac{5}{4}t} dt$

$e^{\frac{5}{4}t} q = 8 e^{5/4 t} + C$

$0 = 8 + C$

$C = -8$

$e^{\frac{5}{4}t} q = 8 e^{5/4 t} - 8$

$q(t) = 8 - 8 e^{-5/4 t}$

$i(t) = 10 e^{-\frac{5}{4}t}$

②

$\frac{dX}{dt} = kX(4000 - X)$

$\frac{dX}{dt} = X(4000k - kX)$

$a = 4000k \quad b = k$

6/

$X(t) = \frac{4000k(1)}{k(1) + (4000k - k(1))e^{-4000kt}}$

$X(t) = \frac{4000}{1 + 3999 e^{-4000kt}}$

$120 = \frac{4000}{1 + 3999 e^{-4000k(10)}}$

$120 + 479880 e^{-40000(10)k} = 4000$

$e^{-40000k} = \frac{3880}{479880}$

$-40000k = \ln \frac{3880}{479880}$

$k = -\frac{1}{40000} \ln \frac{3880}{479880}$

$X(t) = \frac{4000}{1 + 3999 e^{-4000\left(-\frac{1}{40000} \ln \frac{3880}{479880}\right) t}}$

$= \frac{4000}{1 + 3999 e^{\left(\frac{1}{10} \ln \frac{3880}{479880}\right) 25}}$

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(3) $\dfrac{dX}{dt} \propto \left(60-\frac{3}{5}x\right)\left(20-\frac{2}{5}x\right)$

$$\frac{dx}{dt} = k(100-x)(50-x)$$

$$\frac{1}{(100-x)(50-x)} = \frac{A}{100-x} + \frac{B}{50-x}$$

$$1 = A(50-x) + B(100-x)$$

$x = 50 \qquad B = 1/50$

$x = 100 \qquad A = -1/50$

$$\frac{dx}{(100-x)(50-x)} = k\,dt$$

$$\left(\frac{-\frac{1}{50}}{100-x} + \frac{\frac{1}{50}}{50-x}\right)dx = k\,dt$$

$$\frac{1}{50}\ln|100-x| - \frac{1}{50}\ln|50-x| = kt + C_1$$

$$\frac{1}{50}\ln\left|\frac{100-x}{50-x}\right| = kt + C_1$$

$$\ln\left|\frac{100-x}{50-x}\right| = 50kt + C_2$$

7/

$$\frac{100-x}{50-x} = Ce^{50kt}$$

$X(0)=0 \qquad 2 = C$

$$\frac{100-x}{50-x} = 2e^{50kt}$$

$x(5)=10$

$$\frac{90}{40} = 2e^{250k}$$

$$\frac{9}{4} = 2e^{250k}$$

$$\frac{9}{8} = e^{250k}$$

$$\frac{1}{250}\ln\frac{9}{8} = k$$

$$\frac{100-x}{50-x} = 2e^{50\left(\frac{1}{250}\ln\frac{9}{8}\right)t}$$

$$\frac{100-x}{50-x} = 2e^{\left(\frac{1}{5}\ln\frac{9}{8}\right)t}$$

$$\frac{100-x}{50-x} = 2e^{\left(\frac{1}{5}\ln\frac{9}{8}\right)10}$$

$$\frac{100-x}{50-x} = 2\left(\frac{9}{8}\right)^2$$

$$100-x = 2\left(\frac{9}{8}\right)^2 50 - 2\left(\frac{9}{8}\right)^2 x$$

$$2\left(\frac{9}{8}\right)^2 x - x = 2\left(\frac{9}{8}\right)^2 50 - 100$$

$$x\left[2\left(\frac{9}{8}\right)^2 - 1\right] = 2\left(\frac{9}{8}\right)^2 50 - 100$$

$$x = \frac{2\left(\frac{9}{8}\right)^2 50 - 100}{2\left(\frac{9}{8}\right)^2 - 1}$$

$$x = \frac{100\left[\left(\frac{9}{8}\right)^2 - 1\right]}{2\left(\frac{9}{8}\right)^2 - 1}$$

$$x = 17.3g$$

8/ 50g limiting amount of C.

④

$$\begin{vmatrix} 4x^2-3x+5 & 2x^2+4x & 3x^2-1 \\ 8x-3 & 4x+4 & 6x \\ 8 & 4 & 6 \end{vmatrix}$$

$$= 8\left[(2x^2+4x)(6x) - (3x^2-1)(4x+4)\right]$$
$$-4\left[6x(4x^2-3x+5) - (3x^2-1)(8x-3)\right]$$
$$+6\left[(4x^2-3x+5)(4x+4) - (2x^2+4x)(8x-3)\right]$$

$$= 8\left[12x^3+24x^2 - (12x^3+12x^2-4x-4)\right]$$
$$-4\left[24x^3-18x^2+30x - (24x^3-9x^2-8x+3)\right]$$
$$+6\left[(16x^3+16x^2-12x^2-12x+20x+20) - (16x^3-6x^2+32x^2-12x)\right]$$

$$= 8\left[12x^2+4x+4\right] - 4\left[-9x^2+38x-3\right] + 6\left[-22x^2+20x+20\right]$$

$$= 96x^2+32x+32+36x^2-152x+12-132x^2+120x+120$$

$$= 164 \neq 0$$

∴ They are linearly independent

3)

$$\begin{vmatrix} 3x^2+2x-4 & 5x+1 & 6x^2+2 \\ 6x+2 & 5 & 12x \\ 6 & 0 & 12 \end{vmatrix}$$

$$= 6\left[12x(5x+1) - 5(6x^2+2)\right] + 12\left[5(3x^2+2x-4) - (6x+2)(5x+1)\right]$$

$$= 6\left[60x^2+12x-30x^2-10\right] + 12\left[15x^2+10x-20-(30x^2+6x+10x+2)\right]$$

$$= 6\left[30x^2+12x-10\right] + 12\left[(15x^2+10x-20) - (30x^2+16x+2)\right]$$

$$= 6\left[30x^2+12x-10\right] + 12\left[-15x^2-6x-22\right]$$

$$= 180x^2+72x-60-180x^2-72x-264$$

$$= -324 \neq 0$$ ∴ They are linearly independent.

⑤ $f_1(x) = x^3 \quad f_2(x) = x^3 \ln x$

$x^2 y'' - 5xy' + 9y = 0$

$y' = 3x^2 \qquad y' = 3x^2 \ln x + x^3 \cdot \frac{1}{x} = 3x^2 \ln x + x^2$

$y'' = 6x \qquad y'' = 6x \ln x + 3x^2 \cdot \frac{1}{x} + 2x = 6x \ln x + 5x$

$x^2(6x) - 5x(3x^2) + 9(x^3) = 6x^3 - 15x^3 + 9x^3 = 0$

$x^2(6x \ln x + 5x) - 5x(3x^2 \ln x + x^2) + 9(x^3 \ln x) = 6x^3 \ln x + 5x^3 - 15x^3 \ln x - 5x^3 + 9x^3 \ln x = 0$

$$\begin{vmatrix} x^3 & x^3 \ln x \\ 3x^2 & 3x^2 \ln x + x^2 \end{vmatrix} = x^3(3x^2 \ln x + x^2) - 3x^2(x^3 \ln x)$$

$$= 3x^5 \ln x + x^5 - 3x^5 \ln x = x^5 \neq 0$$

∴ They form a fundamental set of solutions

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$f_1(x) = x^4 \quad f_2(x) = x^4 \ln x$

$x^2 y'' - 7xy' + 16y = 0$

$$\begin{vmatrix} x^4 & x^4 \ln x \\ 4x^3 & 4x^3 \ln x + x^3 \end{vmatrix} = x^4(4x^3 \ln x + x^3) - 4x^3(x^4 \ln x) = x^7 \neq 0$$

$y' = 4x^3 \qquad y' = 4x^3 \ln x + x^4 \cdot \frac{1}{x} = 4x^3 \ln x + x^3$

$y'' = 12x^2 \qquad y'' = 12x^2 \ln x + 4x^3 \cdot \frac{1}{x} + 3x^2 = 12x^2 \ln x + 7x^2$

$x^2(12x^2) - 7x(4x^3) + 16(x^4) = 12x^4 - 28x^4 + 16x^4 = 0$

$x^2(12x^2 \ln x + 7x^2) - 7x(4x^3 \ln x + x^3) + 16(x^4 \ln x) = 12x^4 \ln x + 7x^4 - 28x^4 \ln x - 7x^4 + 16x^4 \ln x = 0$

⑥ $x^2 y'' + 9xy' + 12y = 0 \quad y_1 = x^{-2}$

$$y_2 = x^{-2} \int \frac{e^{-\int \frac{9}{x} dx}}{(x^{-2})^2} dx \qquad = x^{-2} \frac{x^{-4}}{-4}$$

6 $$= x^{-2} \int \frac{e^{-9 \ln x}}{x^{-4}} dx \qquad y_2 = x^{-6}$$

$$= x^{-2} \int \frac{x^{-9}}{x^{-4}} dx$$

$$= x^{-2} \int x^{-5} dx$$

$x^2 y'' + 7xy' + 8y = 0 \quad y_1 = x^{-2}$

$$y_2 = x^{-2} \int \frac{e^{-\int \frac{7}{x} dx}}{(x^{-2})^2} dx \qquad = x^{-2} \frac{x^{-2}}{-2}$$

$$= x^{-2} \int \frac{e^{-7 \ln x}}{x^{-4}} dx \qquad y_2 = x^{-4}$$

$$= x^{-2} \int \frac{x^{-7}}{x^{-4}} dx$$

$$= x^{-2} \int x^{-3} dx$$

(7)

$$8y''-10y'-3y=0$$
$$8m^2-10m-3=0$$
$$(4m+1)(2m-3)=0$$
4/
$$m=-1/4,\ 3/2$$
$$y=c_1e^{-1/4x}+c_2e^{3/2x}$$

$$6y''-7y'-3y=0$$
$$6m^2-7m-3=0$$
$$(3m+1)(2m-3)=0$$
$$m=-1/3,\ 3/2$$
$$y=c_1e^{-1/3x}+c_2e^{3/2x}$$

(8)

$$9y''-12y'+4y=0$$
$$9m^2-12m+4=0$$
$$(3m-2)(3m-2)=0$$
4/
$$m=2/3$$
$$y=c_1e^{\frac{2}{3}x}+c_2xe^{\frac{2}{3}x}$$

$$4y''-12y'+9y=0$$
$$4m^2-12m+9=0$$
$$(2m-3)(2m-3)=0$$
$$m=3/2$$
$$y=c_1e^{\frac{3}{2}x}+c_2xe^{\frac{3}{2}x}$$

(9)

$$3y''+2y'+4y=0$$
$$3m^2+2m+4=0$$
$$m=\frac{-2\pm\sqrt{4-4(3)(4)}}{2(3)}$$
4/
$$=\frac{-2\pm\sqrt{4-48}}{6}$$
$$=\frac{-2\pm\sqrt{-44}}{6}$$
$$=-\frac{2}{6}\pm\frac{2i\sqrt{11}}{6}$$
$$=-\frac{1}{3}\pm i\frac{\sqrt{11}}{3}$$
$$y=e^{-\frac{1}{3}x}\left(c_1\cos\frac{\sqrt{11}}{3}x+c_2\sin\frac{\sqrt{11}}{3}x\right)$$

$$4y''+3y'+2y=0$$
$$4m^2+3m+2=0$$
$$m=\frac{-3\pm\sqrt{9-4(4)(2)}}{2(4)}$$
$$=\frac{-3\pm\sqrt{9-32}}{8}$$
$$=\frac{-3\pm\sqrt{-23}}{8}$$
$$=-\frac{3}{8}\pm i\frac{\sqrt{23}}{8}$$
$$y=e^{-\frac{3}{8}x}\left(c_1\cos\frac{\sqrt{23}}{8}x+c_2\sin\frac{\sqrt{23}}{8}x\right)$$

(10)

$$y''' + 9y'' - 4y' - 36y = 0$$

$$m^3 + 9m^2 - 4m - 36 = 0$$

$$m^2(m+9) - 4(m+9) = 0$$

4

$$(m+9)(m+2)(m-2) = 0$$

$$y = C_1 e^{-9x} + C_2 e^{-2x} + C_3 e^{2x}$$

(11)

$$y'' - 2y' - 15y = 8e^{5x}$$

$$m^2 - 2m - 15 = 0$$

$$(m-5)(m+3) = 0$$

$$y_c = C_1 e^{5x} + C_2 e^{-3x}$$

$$y_p = Axe^{5x}$$

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$$y_p' = Ae^{5x} + 5Axe^{5x}$$

$$y_p'' = 5Ae^{5x} + 5Ae^{5x} + 25Axe^{5x}$$

$$= 10Ae^{5x} + 25Axe^{5x}$$

$$(10Ae^{5x} + 25Axe^{5x}) - 2(Ae^{5x} + 5Axe^{5x})$$

$$-15(Axe^{5x}) = 8e^{5x}$$

$$8Ae^{5x} = 8e^{5x}$$

$$8A = 8$$

$$A = 1$$

$$y_p = xe^{5x}$$

$$y = C_1 e^{5x} + C_2 e^{-3x} + xe^{5x}$$

$$y'' - 4y' - 12y = 6e^{-2x}$$

$$m^2 - 4m - 12 = 0$$

$$(m-6)(m+2) = 0$$

$$y_c = C_1 e^{6x} + C_2 e^{-2x}$$

$$y_p = Axe^{-2x}$$

$$y_p' = Ae^{-2x} - 2Axe^{-2x}$$

$$y_p'' = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x}$$

$$= -4Ae^{-2x} + 4Axe^{-2x}$$

$$(-4Ae^{-2x} + 4Axe^{-2x}) - 4(Ae^{-2x} - 2Axe^{-2x})$$

$$-12(Axe^{-2x}) = 6e^{-2x}$$

$$-8Ae^{-2x} = 6e^{-2x}$$

$$-8A = 6$$

$$A = -3/4$$

$$y_p = -\frac{3}{4} xe^{-2x}$$

$$y = C_1 e^{6x} + C_2 e^{-2x} - \frac{3}{4} xe^{-2x}$$

(12)

$$y'' - 8y' + 12y = 6e^{2x} + 2e^{4x}$$

$$m^2 - 8m + 12 = 0$$

$$(m-6)(m-2) = 0$$

$$y_c = c_1 e^{6x} + c_2 e^{2x}$$

$$(D-6)(D-2)y = 6e^{2x} + 2e^{4x}$$

$$(D-6)(D-2)^2(D-4)y = 0$$

7/

$$y = \cancel{c_1 e^{6x}} + c_2 x e^{2x} + \cancel{c_3 e^{2x}} + c_4 e^{4x}$$

$$y_p = Axe^{2x} + Be^{4x}$$

$$y_p' = Ae^{2x} + 2Axe^{2x} + 4Be^{4x}$$

$$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} + 16Be^{4x}$$

$$(4Axe^{2x} + 4Ae^{2x} + 16Be^{4x}) - 8(Ae^{2x} + 2Axe^{2x} + 4Be^{4x}) + 12(Axe^{2x} + Be^{4x}) = 6e^{2x} + 2e^{4x}$$

$$-4Ae^{2x} - 4Be^{4x} = 6e^{2x} + 2e^{4x}$$

$$-4A = 6 \qquad -4B = 2$$

$$A = -\tfrac{3}{2} \qquad B = -\tfrac{1}{2}$$

$$y_p = -\tfrac{3}{2} x e^{2x} - \tfrac{1}{2} e^{4x}$$

$$y = c_1 e^{6x} + c_2 e^{2x} - \tfrac{3}{2} x e^{2x} - \tfrac{1}{2} e^{4x}$$

$$y'' - 6y' + 8y = 4e^{2x} + 2e^{6x}$$

$$m^2 - 6m + 8 = 0$$

$$(m-4)(m-2) = 0$$

$$y_c = c_1 e^{4x} + c_2 e^{2x}$$

$$(D-4)(D-2)y = 4e^{2x} + 2e^{6x}$$

$$(D-4)(D-2)^2(D-6)y = 0$$

$$y = \cancel{c_1 e^{4x}} + \cancel{c_2 e^{2x}} + c_3 x e^{2x} + c_4 e^{6x}$$

$$y_p = Axe^{2x} + Be^{6x}$$

$$y_p' = Ae^{2x} + 2Axe^{2x} + 6Be^{6x}$$

$$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} + 36Be^{6x}$$

$$= 4Ae^{2x} + 4Axe^{2x} + 36Be^{6x}$$

$$(4Axe^{2x} + 4Ae^{2x} + 36Be^{6x}) - 6(Ae^{2x} + 2Axe^{2x} + 6Be^{6x}) + 8(Axe^{2x} + Be^{6x}) = 4e^{2x} + 2e^{6x}$$

$$-2Ae^{2x} + 8Be^{6x} = 4e^{2x} + 2e^{6x}$$

$$-2A = 4 \qquad 8B = 2$$

$$A = -2 \qquad B = \tfrac{1}{4}$$

$$y_p = -2xe^{2x} + \tfrac{1}{4} e^{6x}$$

$$y = c_1 e^{4x} + c_2 e^{2x} - 2xe^{2x} + \tfrac{1}{4} e^{6x}$$

(13) a. $(D-4)^3 D^4$

3/

b. $(D^2 - 4D + 29)(D^2 - 8D + 25)^2$

3/

a. $(D-5)^4 D^3$

b. $(D^2 - 10D + 29)(D^2 - 8D + 25)^2$