MATH 290 – EXAM #2
Winter Session 2019

Name: KEY

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(6 points) *1.* Use Euler's method with the following function and given step size to find y_1, y_2, and y_3. Use at least four decimal places for your answers.

$$y' = xy + y^2, \quad y(1) = 2, \quad h = 0.1$$

$y_1 = 2 + 0.1\left((1)(2) + (2)^2\right)$
$= 2 + 0.1(6)$
$= 2.6$ $\qquad (1.1, 2.6)$

$y_2 = 2.6 + 0.1\left((1.1)(2.6) + (2.6)^2\right)$ $\qquad (1.2, 3.562)$
$= 3.562$

$y_3 = 3.562 + 0.1\left((1.2)(3.562) + (3.562)^2\right)$
$= 5.2582244$ $\qquad (1.3, 5.2582244)$

(9 points) 2. Consider a 20-volt electromotive force that is applied to an *RL*-series circuit in which the resistance is 4 ohms and the inductance is 0.1 henry. Find the current $i(t)$ ~~on the capacitor~~ if $i(0)=0$.

$$L\frac{di}{dt}+Ri=E(t)$$

$$0.1\frac{di}{dt}+4i=20$$

$$\frac{di}{dt}+40i=200$$

$$\mu(t)=e^{\int 40\,dt}=e^{40t}$$

$$\frac{d}{dt}\left[e^{40t}i\right]=200e^{40t}$$

$$e^{40t}i=\int 200e^{40t}\,dt$$

$$e^{40t}i=5e^{40t}+C$$

$$i=5+Ce^{-40t}$$

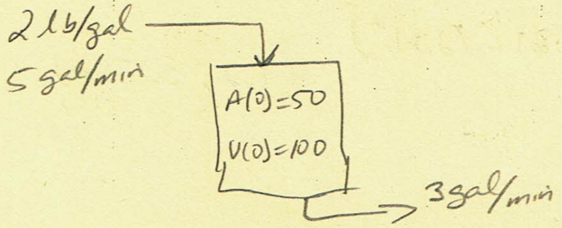
$i(0)=0$

$$0=5+C$$

$$C=-5$$

$$i=5-5e^{-40t}$$

(9 points) 3. A tank initially contains 100 gallons of a solution in which 50 lb of salt are dissolved. A solution containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The well-stirred mixture flows out of the tank at the rate of 3 gal/min. Find the concentration of the salt in the tank 20 minutes after the process starts. Round your answer to the nearest tenth.



$$\frac{dV}{dt}=5-3=2 \qquad V(0)=100$$

$$dV=2dt \qquad 100=C_1$$

$$V=2t+C_1 \qquad V=2t+100$$

$$\frac{dA}{dt}=\frac{2\text{ lb}}{\text{gal}}\cdot\frac{5\text{ gal}}{\text{min}}-\frac{A}{V}\frac{\text{lb}}{\text{gal}}\cdot\frac{3\text{ gal}}{\text{min}}$$

$$\frac{dA}{dt}=10-\frac{3}{2t+100}A$$

$$\frac{dA}{dt}+\frac{3}{2t+100}A=10$$

$$\mu(t)=e^{\int\frac{3}{2t+100}dt}=e^{\frac{3}{2}\ln|2t+100|}$$

$$=(2t+100)^{3/2}$$

$$\frac{d}{dt}\left[(2t+100)^{3/2}A\right]=10(2t+100)^{3/2}$$

$$(2t+100)^{3/2}A=10\int(2t+100)^{3/2}dt$$

$$(2t+100)^{3/2}A=5\cdot\frac{2}{5}(2t+100)^{5/2}+C_2$$

$$(2t+100)^{3/2}A=2(2t+100)^{5/2}+C_2$$

$A(0)=50$

$$100^{3/2}(50)=2(100)^{5/2}+C_2$$

$$10^3(50)-2(10)^5=C_2$$

$$C_2=-150000$$

$$(2t+100)^{3/2}A=2(2t+100)^{5/2}-150000$$

$t=20$

$$(140)^{3/2}A=2(140)^{5/2}-150000$$

$$A\doteq 189.447\text{ lb}$$

$$V=2(20)+100=140\text{ gal}$$

concentration: $\dfrac{189.447\text{ lb}}{140\text{ gal}}\doteq 1.35\text{ lb/gal}$

4. A flammable substance whose initial temperature is 50°F is inadvertently placed in a hot oven whose temperature is 450°F. After 20 minutes, the substance's temperature is 150°F.

(7 points) a. Find the temperature of the substance after 40 minutes.

$$\frac{dT}{dt} = k(T - T_m)$$

$$\frac{dT}{T - T_m} = k\,dt$$

$$\ln|T - T_m| = kt + C_1$$

$$T - T_m = Ce^{kt}$$

$$T(0) = 50^\circ F \quad T_m = 450^\circ F$$

$$50 - 450 = Ce^{k(0)}$$

$$-400 = C$$

$$T - 450 = -400e^{kt}$$

$$T(20) = 150^\circ F$$

$$150 - 450 = -400e^{20k}$$

$$-300 = -400e^{20k}$$

$$\frac{3}{4} = e^{20k}$$

$$\ln\frac{3}{4} = 20k$$

$$k = \frac{1}{20}\ln\frac{3}{4}$$

$$T = -400e^{(\frac{1}{20}\ln\frac{3}{4})t} + 450$$

$$t = 40$$

$$T = -400e^{(\frac{1}{20}\ln\frac{3}{4})40} + 450$$

$$= -400e^{2\ln\frac{3}{4}} + 450$$

$$= -400\left(\frac{3}{4}\right)^2 + 450$$

$$= 225^\circ F$$

(2 points) b. Assuming the substance ignites when its temperature reaches 350°F, find the time of combustion.

$$350 = -400e^{(\frac{1}{20}\ln\frac{3}{4})t} + 450$$

$$-100 = -400e^{(\frac{1}{20}\ln\frac{3}{4})t}$$

$$\frac{1}{4} = e^{(\frac{1}{20}\ln\frac{3}{4})t}$$

$$\ln\frac{1}{4} = \frac{1}{20}\ln\frac{3}{4}t$$

$$t = \frac{20\ln\frac{1}{4}}{\ln\frac{3}{4}} \doteq 96 \text{ minutes.}$$

(5 points) 5. Construct a mathematical model for a radioactive series of four elements *W, X, Y*, and *Z* where $W \to X \to Y \to Z$, that is, *Z* is a stable element. The decay constants for *W, X*, and *Y* are λ_1, λ_2, and λ_3, respectively. The amount of elements *W, X, Y*, and *Z* are $w(t)$, $x(t)$, $y(t)$, and $z(t)$, respectively. The model is to express the rate of change of $w(t)$, $x(t)$, $y(t)$, and $z(t)$ with respect to time.

$$\frac{dw}{dt} = -\lambda_1 w$$

$$\frac{dx}{dt} = \lambda_1 w - \lambda_2 x$$

$$\frac{dy}{dt} = \lambda_2 x - \lambda_3 y$$

$$\frac{dz}{dt} = \lambda_3 y$$

6. Two chemicals A and B are combined to form a chemical C. The rate of reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 40 grams of A and 50 grams of B, and for each gram of B, 2 grams of A are used. It is observed that 10 grams of C are formed in 5 minutes.

(7 points) *a.* How much of C is formed in 20 minutes?

$\frac{dx}{dt} = k\left(40 - \frac{2}{3}x\right)\left(50 - \frac{1}{3}x\right)$

$\frac{dx}{dt} = k(60-x)(150-x)$

$\frac{dx}{(60-x)(150-x)} = k\,dt$

$\frac{1}{(60-x)(150-x)} = \frac{A}{60-x} + \frac{B}{150-x}$

$1 = A(150-x) + B(60-x)$

$x = 60 \quad A = 1/90$

$x = 150 \quad B = -1/90$

$\left(\frac{1/90}{60-x} - \frac{1/90}{150-x}\right)dx = k\,dt$

$-\frac{1}{90}\ln|60-x| + \frac{1}{90}\ln|150-x| = kt + C$

$\frac{1}{90}\ln\left|\frac{150-x}{60-x}\right| = kt + C$

$t=0, \; x=0$

$\frac{1}{90}\ln\left(\frac{150}{60}\right) = C$

$C = \frac{1}{90}\ln\frac{5}{2}$

$\frac{1}{90}\ln\left|\frac{150-x}{60-x}\right| = kt + \frac{1}{90}\ln\frac{5}{2}$

$t=5, \; x=10$

$\frac{1}{90}\ln\frac{140}{50} = 5k + \frac{1}{90}\ln\frac{5}{2}$

$\frac{1}{90}\ln\frac{14}{5} - \frac{1}{90}\ln\frac{5}{2} = 5k$

$5k = \frac{1}{90}\ln\frac{14/5}{5/2} \qquad k = \frac{1}{450}\ln\frac{28}{25}$

$\frac{1}{90}\ln\left|\frac{150-x}{60-x}\right| = \left(\frac{1}{450}\ln\frac{28}{25}\right)t + \frac{1}{90}\ln\frac{5}{2}$

multiply by 90

$\ln\left|\frac{150-x}{60-x}\right| = \left(\frac{1}{5}\ln\frac{28}{25}\right)t + \ln\frac{5}{2}$

$t=20$ Find x

$\ln\left|\frac{150-x}{60-x}\right| = \left(\frac{1}{5}\ln\frac{28}{25}\right)20 + \ln\frac{5}{2}$

$\ln\left|\frac{150-x}{60-x}\right| = 4\ln\frac{28}{25} + \ln\frac{5}{2}$

$\ln\left|\frac{150-x}{60-x}\right| = \ln\left(\frac{28}{25}\right)^4 \cdot \frac{5}{2}$

$\frac{150-x}{60-x} = \left(\frac{28}{25}\right)^4 \cdot \frac{5}{2}$

$\frac{150-x}{60-x} = 3.9337984$

$150 - x = 236.027 - 3.9337x$

$x \doteq 29.3\text{ g}$

(2 points) *b.* What is the limiting amount of C?

$x < 60\text{ g}$

$\therefore$ The limiting amount is 60g.

(5 points) 7. Verify that $y_1(x) = x^3$ and $y_2(x) = x^4$ form a fundamental set of solutions to the differential equation $x^2y'' - 6xy' + 12y = 0$ on the interval $(0,\infty)$.

Wronskian:

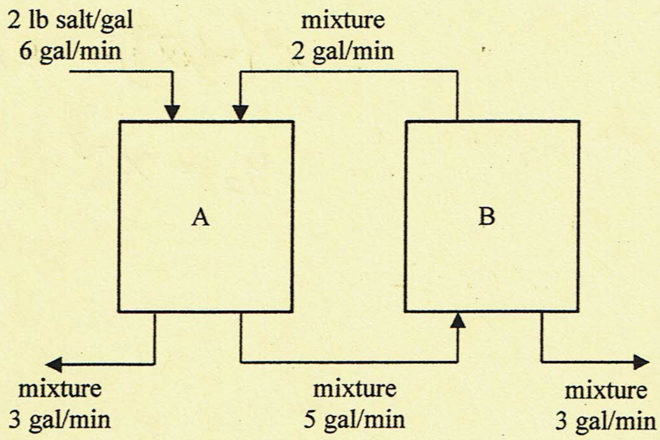
$$\begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = 4x^6 - 3x^6 = x^6 \neq 0$$

on $(0,\infty)$

$\therefore$ x^3 and x^4 are linearly independent,

$\therefore$ They form a fundamental set of solutions to the differential equation.

8. Consider the two tanks shown in the given figure. Initially, tank A contains 200 gallons of water in which 50 pounds of salt is dissolved and tank B contains 200 gallons of pure water. Liquid is pumped in and out of the tanks as indicated in the figure. The mixture exchanged between the two tanks and the mixture pumped out of tank B are assumed to be well-stirred.



(5 points) *a.* Construct a mathematical model using a system of differential equations that describes rate of change of the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks *A* and *B*, respectively, at time *t*.

$$\frac{dx_1}{dt} = \frac{2\,lb}{gal}\cdot\frac{6\,gal}{min} + \frac{x_2\,lb}{gal}\cdot\frac{2\,gal}{min} - \frac{x_1\,lb}{gal}\cdot\frac{3\,gal}{min} - \frac{x_1\,lb}{gal}\cdot\frac{5\,gal}{min}$$

$$\frac{dx_2}{dt} = \frac{x_1\,lb}{gal}\cdot\frac{5\,gal}{min} - \frac{x_2\,lb}{gal}\cdot\frac{2\,gal}{min} - \frac{x_2\,lb}{gal}\cdot\frac{3\,gal}{min}$$

(2 points) *b.* State the initial conditions of tank *A* and tank B.

$x_1(0) = 50\ lbs$ $\qquad x_2(0) = 0\ lbs.$

(5 points) *9.* Determine if the following set of functions is linearly independent or linearly dependent.

$$f_1(x) = 3 + x,\ f_2(x) = x,\ f_3(x) = x^2$$

$$\begin{vmatrix} 3+x & x & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (3+x)(2-0) - x(2-0) + x^2(0-0)$$

$$= 6 + 2x - 2x = 6 \neq 0 \text{ on } (-\infty, \infty)$$

$\therefore$ $3+x$, x, x^2 are linearly independent.

(9 points) 10. Given that $y_1(x) = x^2$ is a solution to $x^2y'' + 2xy' - 6y = 0$. Use reduction of order to find a second solution, $y_2(x)$.

$$y'' + \frac{2}{x} - \frac{6}{x^2} = 0$$

$$y_2 = x^2 \int \frac{e^{-\int \frac{2}{x}dx}}{(x^2)^2}\,dx$$

$$= x^2 \int \frac{e^{-2\ln|x|}}{x^4}\,dx$$

$$= x^2 \int \frac{x^{-2}}{x^4}\,dx$$

$$= x^2 \int x^{-6}\,dx$$

$$= x^2\left(-\frac{1}{5}x^{-5}\right) = -\frac{1}{5}x^{-3}$$

$$y_2 = x^{-3}$$

11. Solve the following differential equations.

(4 points) a. $y'' + 12y' + 36y = 0$

$$m^2 + 12m + 36 = 0$$

$$(m+6)^2 = 0$$

$$m = -6$$

$$y_1 = e^{-6x} \quad y_2 = xe^{-6x}$$

$$y = c_1 e^{-6x} + c_2 x e^{-6x}$$

(4 points) b. $2y'' - 6y' + 7y = 0$

$$2m^2 - 6m + 7 = 0$$

$$m = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(7)}}{2(2)} = \frac{6 \pm \sqrt{36-56}}{4} = \frac{6 \pm \sqrt{-20}}{4} = \frac{6 \pm 2i\sqrt{5}}{4}$$

$$= \frac{3}{2} \pm \frac{i\sqrt{5}}{2}$$

$$y = e^{\frac{3}{2}x}\left(c_1 \cos \frac{\sqrt{5}}{2}x + c_2 \sin \frac{\sqrt{5}}{2}x\right)$$

(4 points) c. $3y''-10y'+8y=0$

$3m^2-10m+8=0$

$(3m-4)(m-2)=0$

$m=4/3,\ 2$

$y=C_1e^{4/3x}+C_2e^{2x}$

(4 points) d. $y'''-2y''-8y'=0$

$m^3-2m^2-8m=0$

$m(m^2-2m-8)=0$

$m(m-4)(m+2)=0$

$m=0,4,-2$

$y=C_1+C_2e^{4x}+C_3e^{-2x}$

12. Given the following general solution to the given differential equation.

$$y=c_1e^x\cos x+c_2e^x\sin x; \qquad y''-2y'+4y=0$$

Determine if the following boundary conditions yield no solution, one solution, or infinitely many solutions.

(2 points) a. $y(0)=0,\ y(\pi)=0$

$0=C_1$
$0=C_1e^{\pi}$ $\Rightarrow$ $C_1=0$, $C_1=0$ C_2 is free

infinitely many solutions

(2 points) b. $y(0)=1,\ y(\pi)=-1$

$1=C_1$
$-1=C_1e^{\pi}$ $\Rightarrow$ $C_1=1$, $C_1=-\frac{1}{e^{\pi}}$ > contradiction

no solution

(2 points) c. $y(0)=1,\ y(\pi/2)=1$

$1=C_1$

$1=C_2e^{\pi/2}\Rightarrow C_2=e^{-\pi/2}$ one solution

13. Given the following general solution to the given differential equation.

$$y = c_1e^{5x} + c_2e^{x} + 3e^{2x}; \quad y'' - 6y' + 5y = -9e^{2x}$$

(3 points) a. Verify that y is a solution to the differential equation.

$y' = 5c_1e^{5x} + c_2e^{x} + 6e^{2x}$ $\quad y'' = 25c_1e^{5x} + c_2e^{x} + 12e^{2x}$

$(25c_1e^{5x} + c_2e^{x} + 12e^{2x}) - 6(5c_1e^{5x} + c_2e^{x} + 6e^{2x}) + 5(c_1e^{5x} + c_2e^{x} + 3e^{2x})$

$= 25c_1e^{5x} + c_2e^{x} + 12e^{2x} - 30c_1e^{5x} - 6c_2e^{x} - 36e^{2x} + 5c_1e^{5x} + 5c_2e^{x} + 15e^{2x}$

$= -9e^{2x}$

(1 point) b. What is the $y = c_1e^{5x} + c_2e^{x}$ part called in the solution?

Complementary solution

(1 point) c. What is the $y = 3e^{2x}$ part called in the solution?

particular solution

(1 point) d. A differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is called what kind of equation?

homogeneous equation

(1 point) e. A differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where $g(x)$ is not identically zero is called what kind of equation?

nonhomogeneous equation.