MATH 290 – EXAM #2
Winter Session 2020

Name: KEY

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(9 points) 1. Consider a 30-volt electromotive force that is applied to an *RL*-series circuit in which the resistance is 10 ohms and the inductance is 0.2 henry. Find the current $i(t)$ on the capacitor if $i(0)=0$.

$$L\frac{di}{dt} + Ri = E(t)$$

$$0.2\frac{di}{dt} + 10i = 30$$

$$\frac{di}{dt} + 50i = 150$$

$$\mu(t) = e^{\int 50\,dt} = e^{50t}$$

$$e^{50t} i = 150\int e^{50t}\,dt$$

$$e^{50t} i = 3e^{50t} + C$$

$$i = 3 + Ce^{-50t}$$

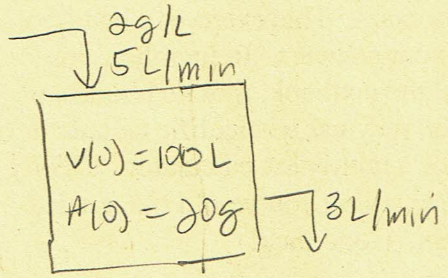
$i(0)=0$

$$0 = 3 + C$$

$$C = -3$$

$$i(t) = 3 - 3e^{-50t}$$

(9 points) 2. A tank initially contains 100 L of a solution in which 20 g of salt is dissolved. A solution with a salt concentration of 2 g/L is added at a rate of 5 L/min. The solution is kept well mixed and is drained from the tank at a rate of 3 L/min. Find the concentration of the solution in the tank after 20 minutes.



$\frac{dV}{dt} = 5-3$

$dV = 2dt$

$V = 2t + C_1$

$V(0) = 100$

$100 = 2(0) + C_1$

$C_1 = 100$

$V = 2t + 100$

$\frac{dA}{dt} = \left(2\frac{g}{L}\right)\left(5\frac{L}{min}\right) - \left(\frac{A}{V}\frac{g}{L}\right)\left(3\frac{L}{min}\right)$

$\frac{dA}{dt} = 10 - \frac{3}{2t+100}A$

$\frac{dA}{dt} + \frac{3}{2t+100}A = 100$

$\mu(t) = e^{\int \frac{3}{2t+100}dt} = e^{\frac{3}{2}\int \frac{1}{t+50}dt} = e^{\frac{3}{2}\ln|t+50|} = (t+50)^{3/2}$

$(t+50)^{3/2}A = 10\int (t+50)^{3/2}dt$

$(t+50)^{3/2}A = 10 \cdot \frac{2}{5}(t+50)^{5/2} + C_2$

$(t+50)^{3/2}A = 4(t+50)^{5/2} + C_2$

$A(0) = 20$

$50^{3/2}(20) = 4(50)^{5/2} + C_2$

$50^{3/2}(20) - 4(50)^{5/2} = C_2$

$50^{3/2}(20 - 200) = C_2$

$50^{3/2}(-180) = C_2$

$(t+50)^{3/2}A = 4(t+50)^{5/2} - 50^{3/2}180$

$t=20$ $\quad 70^{3/2}A = 4(70)^{5/2} - 50^{3/2}180 \quad V = 140\text{ L}$

$A = \frac{4(70)^{5/2} - 50^{3/2}180}{70^{3/2}}\text{ g}$

concentration $= \dfrac{\frac{4(70)^{5/2} - 50^{3/2}180}{70^{3/2}}\text{ g}}{140\text{ L}}$

$\doteq 1.22\ g/L$

3. Two chemicals A and B are combined to form a chemical C. The rate of reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 60 grams of A and 20 grams of B, and for each gram of B, 4 grams of A are used. It is observed that 10 grams of C are formed in 5 minutes.

(9 points) *a.* How much of C is formed in 15 minutes?

$$\frac{dx}{dt} \propto \left(60-\tfrac{4}{5}x\right)\left(20-\tfrac{1}{5}x\right)$$

$$\frac{dx}{dt} = k(75-x)(100-x)$$

$$\frac{dx}{(75-x)(100-x)} = k\,dt$$

$$\frac{1}{(75-x)(100-x)} = \frac{A}{75-x} + \frac{B}{100-x}$$

$$1 = A(100-x) + B(75-x)$$

$x = 75 \quad A = 25$

$x = 100 \quad B = -25$

$$\left(\frac{25}{75-x} - \frac{25}{100-x}\right) dx = k\,dt$$

$$-25\ln|75-x| + 25\ln|100-x| = kt + C$$

$$\ln\left|\frac{100-x}{75-x}\right| = 25kt + 25C$$

$$\frac{100-x}{75-x} = C_1 e^{25kt}$$

$x(0) = 0$

$$\frac{100}{75} = C_1 = \frac{4}{3} \qquad \frac{100-x}{75-x} = \frac{4}{3}e^{25kt}$$

$x(5) = 10$

$$\frac{100-10}{75-10} = \frac{90}{65} = \frac{4}{3}e^{25k(5)}$$

$$\frac{18}{13}\cdot\frac{3}{4} = e^{125k}$$

$$\frac{27}{26} = e^{125k}$$

$$\frac{1}{125}\ln\frac{27}{26} = k$$

$$\frac{100-x}{75-x} = \frac{4}{3}e^{25\left(\frac{1}{125}\ln\frac{27}{26}\right)t}$$

$t = 15$

$$\frac{100-x}{75-x} = \frac{4}{3}e^{\left(\frac{1}{5}\ln\frac{27}{26}\right)t}$$

$$\frac{100-x}{75-x} = \frac{4}{3}\left(\frac{27}{26}\right)^{\frac{1}{5}(15)}$$

$$100-x = (1.49317)(75-x)$$

$$100-x = (1.49317)75 - (1.49317)x$$

$$1.49317x - x = (1.49317)75 - 100$$

$$x = \frac{(1.49317)75-100}{1.49317-1}$$

$$x \doteq 24.3\text{ g}$$

(3 points) *b.* What is the limiting amount of C?

Limiting amount of C is 75 g

4. An object whose initial temperature is 60°F is placed in a hot oven whose temperature is 400°F. After 20 minutes, the object's temperature is 180°F.

(7 points) *a.* Find the temperature of the object after 30 minutes.

$T = T_m + Ce^{kt}$

$T = 400 + Ce^{kt}$

$T(0) = 60$

$60 = 400 + C$

$-340 = C$

$T = 400 - 340e^{kt}$

$T(20) = 180$

$180 = 400 - 340e^{k(20)}$

$-220 = -340e^{20k}$

$\frac{11}{17} = e^{20k}$

$\frac{1}{20}\ln\frac{11}{17} = k$

$T = 400 - 340e^{\left(\frac{1}{20}\ln\frac{11}{17}\right)t}$

$t = 30$

$T = 400 - 340e^{\left(\frac{1}{20}\ln\frac{11}{17}\right)30}$

$T \doteq 223.03°F$

(3 points) *b.* Find the time when the object reaches a temperature of 350°F.

$350 = 400 - 340e^{\left(\frac{1}{20}\ln\frac{11}{17}\right)t}$

$-50 = -340e^{\left(\frac{1}{20}\ln\frac{11}{17}\right)t}$

$\frac{5}{34} = e^{\left(\frac{1}{20}\ln\frac{11}{17}\right)t}$

$\ln\frac{5}{34} = \left(\frac{1}{20}\ln\frac{11}{17}\right)t$

$t = \dfrac{\ln\frac{5}{34}}{\frac{1}{20}\ln\frac{11}{17}}$

$t = 88.07$ minutes.

(6 points) 5. Verify that $y_1(x) = \sin 4x$ and $y_2(x) = \cos 4x$ form a fundamental set of solutions to the differential equation $y'' + 16y = 0$ on the interval $(-\infty, \infty)$.

$y = c_1 \sin 4x + c_2 \cos 4x$

$y' = 4c_2 \cos 4x - 4c_2 \sin 4x$

$y'' = -16c_2 \sin 4x - 16c_2 \cos 4x$

$(-16c_2 \sin 4x - 16c_2 \cos 4x) + 16(c_1 \sin 4x + c_2 \cos 4x)$

$= 0$

$\therefore$ y_1 and y_2 are solutions to the DE.

$$\begin{vmatrix} \sin 4x & \cos 4x \\ 4\cos 4x & -4\sin 4x \end{vmatrix} = -4\sin^2 4x - 4\cos^2 4x$$

$$= -4 \neq 0$$

$\therefore$ y_1 and y_2 are linearly independent

$\therefore$ y_1 and y_2 form a fundamental set of solutions.

(6 points) 6. Determine if the following set of functions is linearly independent or linearly dependent.

$$f_1(x)=3+4x+2x^2,\ f_2(x)=2x-x^2,\ f_3(x)=5+3x$$

$$\begin{vmatrix} 3+4x+2x^2 & 2x-x^2 & 5+3x \\ 4+4x & 2-2x & 3 \\ 4 & -2 & 0 \end{vmatrix} = 4\left(3(2x-x^2)-(5+3x)(2-2x)\right) - (-2)\left(3(3+4x+2x^2)-(5+3x)(4+4x)\right)+0$$

$$=4(6x-3x^2-10+4x+6x^2)+2(9+12x+6x^2-20-32x-12x^2)$$

$$=4(3x^2+10x-10)+2(-6x^2-20x-11)$$

$$=12x^2+40x-40-12x^2+40x-22=-62\neq 0$$ ∴ They are linearly independent

(9 points) 7. Suppose a student carrying a flu virus returns to an isolated college campus of 3000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not inflected, determine the number of infected students after 20 days if it is further observed that after 8 days $x(8)=150$.

$$\frac{dx}{dt}\propto x(3000-x)$$

$$\frac{dx}{dt}=kx(3000-x)$$

$$\frac{dx}{dt}=x(3000k-kx)$$

$a=3000k$, $b=k$ $P_0=1$

$$P=\frac{aP_0}{e^{-at}(a-bP_0)+bP_0}$$

$$P=\frac{3000k}{e^{-3000kt}(3000k-k)+k}$$

$$P=\frac{3000}{2999e^{-3000kt}+1}$$

$x(8)=150$

$$150=\frac{3000}{2999e^{-3000k(8)}+1}$$

$$2999e^{-24000k}+1=20$$

$$2999e^{-24000k}=19$$

$$e^{-24000k}=\frac{19}{2999}$$

$$k=-\frac{1}{24000}\ln\frac{19}{2999}$$

$$P=\frac{3000}{2999e^{(\frac{1}{8}\ln\frac{19}{2999})t}+1}$$

$t=20$

$$P=\frac{3000}{2999\left(\frac{19}{2999}\right)^{5/2}+1}\doteq 2972 \text{ students,}$$

(8 points) 8. Given that $y_1(x) = x^2$ is a solution to $x^2 y'' + 3xy' - 8y = 0$. Use reduction of order to find a second solution, $y_2(x)$.

$y_2 = u_1 y_1 = u_1 x^2$

$y_2' = u_1' x^2 + u_1(2x)$

$y_2'' = u_1'' x^2 + 4xu_1' + 2u_1$

$x^2(u_1'' x^2 + 4xu_1' + 2u_1) + 3x(u_1' x^2 + 2xu_1) - 8(u_1 x^2) = 0$

$u_1'' x^4 + 4x^3 u_1' + 2u_1 x^2 + 3x^3 u_1' + 6x^2 u_1 - 8u_1 x^2 = 0$

$u_1'' x^4 + u_1' 7x^3 = 0$

let $w = u_1'$, $w' = u_1''$

$w' x^4 + 7x^3 w = 0$

$\frac{dw}{dx} = -\frac{7}{x} w$

$\frac{dw}{w} = -\frac{7}{x} dx$

$\ln|w| = -7\ln|x| + C_1$

$w = C_2 x^{-7}$

$u_1' = C_2 x^{-7}$

$u_1 = -\frac{1}{6} C_2 x^{-6} + C_3$

$y_2 = -\frac{1}{6} C_2 x^{-4} + C_3 x^2$

$y_2 = x^{-4}$

9. Given the following general solution to the given differential equation.

$$y = c_1 e^x \cos x + c_2 e^x \sin x; \qquad y'' - 2y' + 4y = 0$$

Determine if the following boundary conditions yield no solution, one solution, or infinitely many solutions.

(3 points) a. $y(0) = 0,\ y(\pi) = 0$

$0 = C_1$

$0 = C_1 e^{\pi}(-1) \Rightarrow C_1 = 0$

C_2 is free

infinite # of solutions

(3 points) b. $y(0) = 1,\ y(\pi) = -1$

$1 = C_1$

$-1 = -C_1 e^{\pi}(-1) \Rightarrow C_1 = e^{-\pi}$ > contradiction

No solution

(3 points) c. $y(0) = 1,\ y(\pi/2) = 1$

$1 = C_1$

$1 = C_2 e^{\pi/2}$

$C_2 = e^{-\pi/2}$

one solution

10. Solve the following differential equations.

(4 points) *a.* $2y''-4y'+3y=0$

$2m^2-4m+3=0$

$m=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(3)}}{2(2)}=\frac{4\pm\sqrt{16-24}}{4}=\frac{4\pm\sqrt{-8}}{4}$

$=\frac{4\pm 2i\sqrt{2}}{4}=1\pm i\frac{\sqrt{2}}{2}$ $\quad y=e^x\left(c_1\cos\frac{\sqrt{2}}{2}x+c_2\sin\frac{\sqrt{2}}{2}x\right)$

(4 points) *b.* $9y''+12y'+4y=0$

$9m^2+12m+4=0$

$(3m+2)^2=0$

$m=-2/3$ multiplicity 2 $\quad y=c_1e^{-\frac{2}{3}x}+c_2xe^{-\frac{2}{3}x}$

(4 points) *c.* $8y''-14y'-15y=0$

$8m^2-14m-15=0$

$(4m+3)(2m-5)=0$

$m=-3/4,\ 5/2$

$y=c_1e^{-3/4x}+c_2e^{5/2x}$

(4 points) *d.* $y'''+2y''-24y'=0$

$m^3+2m^2-24m=0$

$m(m^2+2m-24)=0$

$m(m+6)(m-4)=0$

$y=c_1+c_2e^{-6x}+c_3e^{4x}$

(8 points) *11.* Solve the following differential equation by using the superposition method.

$$y'' - 8y' + 12y = 4\sin 2x$$

$y'' - 8y' + 12y = 0$

$m^2 - 8m + 12 = 0$

$(m-6)(m-2) = 0$

$y_c = c_1 e^{6x} + c_2 e^{2x}$

$y_p = A\cos 2x + B\sin 2x$

$y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

$(-4A\cos 2x - 4B\sin 2x) - 8(-2A\sin 2x + 2B\cos 2x) + 12(A\cos 2x + B\sin 2x) = 4\sin 2x$

$(16A + 8B)\sin 2x + (8A - 16B)\cos 2x = 4\sin 2x$

$2(16A + 8B = 4)$

$8A - 16B = 0$

$32A + 16B = 8$

$8A - 16B = 0$

$40A = 8$

$A = 1/5$

$8(\frac{1}{5}) - 16B = 0$

$\frac{8}{5} = 16B$

$\frac{1}{10} = B$

$y_p = \frac{1}{5}\cos 2x + \frac{1}{10}\sin 2x$

$y = c_1 e^{6x} + c_2 e^{2x} + \frac{1}{5}\cos 2x + \frac{1}{10}\sin 2x$