MATH 290 – EXAM #3
Spring Semester 2019

Name: KEY

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(10 points) 1. Solve the following differential equation by using annihilators.

$$y'' - 4y' + 3y = 2e^x + 4x$$

$y'' - 4y' + 3y = 0$

$m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0$

$y_c = c_1 e^{3x} + c_2 e^x$

$(D-3)(D-1)y = 2e^x + 4x$

$D^2(D-1)(D-3)(D-1)y = 0$

$D^2(D-1)^2(D-3)y = 0$

$y = c_1 + c_2 x + \cancel{c_3 e^x} + c_4 x e^x + \cancel{c_5 e^{3x}}$

$y_p = A + Bx + Cxe^x$

$y_p' = B + Ce^x + Cxe^x$

$y_p'' = 2Ce^x + Cxe^x$

$(2Ce^x + Cxe^x) - 4(B + Ce^x + Cxe^x) + 3(A + Bx + Cxe^x) = 2e^x + 4x$

$(3A - 4B) + (3B)x + (-2C)e^x = 2e^x + 4x$

$3A - 4B = 0 \quad 3B = 4 \quad -2C = 2$

$B = 4/3 \quad C = -1$

$3A = \frac{16}{3}$

$A = \frac{16}{9}$

$y_p = \frac{16}{9} + \frac{4}{3}x - xe^x$

$y = c_1 e^{3x} + c_2 e^x + \frac{16}{9} + \frac{4}{3}x - xe^x$

2. Give the annihilator operator for the following functions.

(2 points) *a.* $3x^2e^{4x}+5x^3$

$D^4(D-4)^3$

(2 points) *b.* $e^{5x}\sin 2x$

$D^2+(-2\cdot 5)D+(2^2+5^2)$

$D^2-10D+29$

(10 points) 3. Solve the following equation by using variation of parameters.

$$y''+4y'+4y=e^{-2x}\ln x$$

$m^2+4m+4=0$

$(m+2)^2=0$

$y_c=c_1e^{-2x}+c_2xe^{-2x}$

$y_p=u_1e^{-2x}+u_2xe^{-2x}$

$u_1'e^{-2x}+u_2'xe^{-2x}=0$

$u_1'(-2e^{-2x})+u_2'(e^{-2x}-2xe^{-2x})=e^{-2x}\ln x$

$u_1'+u_2'x=0$

$-2u_1'+(1-2x)u_2'=\ln x$

$\begin{vmatrix}1 & x\\ -2 & (1-2x)\end{vmatrix}=(1-2x)+2x=1$

$u_1'=\dfrac{\begin{vmatrix}0 & x\\ \ln x & 1-2x\end{vmatrix}}{1}=-x\ln x$

$u_1'=\int(-x\ln x)dx$ $\quad u=\ln x \quad dv=-x\,dx$ $\quad du=\frac{1}{x}dx \quad v=-\frac{x^2}{2}$

$=-\frac{x^2}{2}\ln x+\int\frac{1}{2}x\,dx=-\frac{x^2}{2}\ln x+\frac{x^2}{4}$

$u_2'=\dfrac{\begin{vmatrix}1 & 0\\ -2 & \ln x\end{vmatrix}}{1}=\ln x$

$u_2'=\int\ln x\,dx$ $\quad u=\ln x \quad dv=dx$ $\quad du=\frac{1}{x}dx \quad v=x$

$=x\ln x-\int dx=x\ln x-x$

$y_p=(-\frac{x^2}{2}\ln x+\frac{x^2}{4})e^{-2x}+(x\ln x-x)xe^{-2x}$

$=\frac{1}{2}x\ln x-\frac{3}{4}x^2e^{-2x}$

$y=c_1e^{-2x}+c_2xe^{-2x}+(-\frac{x^2}{2}\ln x+\frac{x^2}{4})e^{-2x}+(x\ln x-x)xe^{-2x}=c_1e^{-2x}+c_2xe^{-2x}+\frac{1}{2}x^2\ln x-\frac{3}{4}x^2e^{-2x}$

(10 points) *4.* Solve the following differential equation using any method.

$$y'' - 5y' + 4y = 6\cos x$$

$m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$m = 4, 1$

$y_c = c_1 e^{4x} + c_2 e^x$

$(D-4)(D-1)(D^2+1)y = 0$

$y = c_1 e^{4x} + c_2 e^x + c_3 \cos x + c_4 \sin x$

$y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$(-A\cos x - B\sin x) - 5(-A\sin x + B\cos x) + 4(A\cos x + B\sin x) = 6\cos x$

$(3A - 5B)\cos x + (5A + 3B)\sin x = 6\cos x$

$3(3A - 5B = 6)$

$5(5A + 3B = 0)$

$9A - 15B = 18$

$25A + 15B = 0$

$34A = 18$

$A = 18/34 = 9/17$

$5(\frac{9}{17}) + 3B = 0$

$3B = -\frac{45}{17}$

$B = -\frac{45}{51} = -\frac{15}{17}$

$y = c_1 e^{4x} + c_2 e^x + \frac{9}{17}\cos x - \frac{15}{17}\sin x$

5. Solve the following differential equation.

(4 points) *a.* $x^2 y'' + xy' + 4y = 0$

$m^2 + (1-1)m + 4 = 0$

$m^2 + 4 = 0$

$m = \pm 2i$

$y = c_1 \cos(2\ln x) + c_2 \sin(2\ln x)$

(4 points) *b.* $x^2 y'' + 5xy' + 3y = 0$

$m^2 + (5-1)m + 3 = 0$

$m^2 + 4m + 3 = 0$

$(m+3)(m+1) = 0$

$m = -3, -1$

$y = c_1 x^{-3} + c_2 x^{-1}$

(8 points) *6.* Solve the following differential equation by using variation of parameters.

$$x^2y'' - xy' + y = 2x$$

$m^2+(-1-1)m+1=0$

$m^2-2m+1=0$

$(m-1)^2=0$

$y_c=c_1x+c_2x\ln x$

$y_p=u_1x+u_2x\ln x$

$u_1'x+u_2'x\ln x=0$

$u_1'+u_2'(\ln x+1)=\frac{2}{x}$

$\begin{vmatrix} x & x\ln x \\ 1 & \ln x+1 \end{vmatrix}=$

$x(\ln x+1)-x\ln x=x$

$u_1'=\dfrac{\begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & \ln x+1 \end{vmatrix}}{x}=\frac{-2\ln x}{x}$ $\quad u_1=-\int 2x^{-1}\ln x\,dx$

$u=\ln x \quad du=\frac{1}{x}$

$-\int 2\frac{\ln x}{x}dx=-\int 2u\,du=-u^2=-(\ln x)^2$

$u_2'=\dfrac{\begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix}}{x}=\frac{2}{x}$ $\quad u_2=\int\frac{2}{x}dx=2\ln x$

$y_p=-x(\ln x)^2+2x(\ln x)^2=x(\ln x)^2$

$y=c_1x+c_2x\ln x+x(\ln x)^2$

7. Determine the inverse Laplace transform of the following function.

(3 points) *a.* $\mathscr{L}^{-1}\left\{\frac{4}{s^2}-\frac{6}{s^5}+\frac{1}{s-4}\right\}$ $=L^{-1}\left[4\cdot\frac{1}{s^2}\right]-L^{-1}\left[\frac{6}{4!}\cdot\frac{4!}{s^5}\right]+L^{-1}\left[\frac{1}{s-4}\right]$

$=4t-\frac{1}{4}t^4+e^{4t}$

(3 points) *b.* $\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\}$ $=L^{-1}\left[\frac{2s+5}{s^2+6s+9+34-9}\right]=L^{-1}\left[\frac{2s+5}{(s+3)^2+25}\right]$

$=2L^{-1}\left[\frac{s}{(s+3)^2+25}\right]+L^{-1}\left[\frac{5}{(s+3)^2+25}\right]$

$=2L^{-1}\left[\frac{s+3-3}{(s+3)^2+25}\right]+L^{-1}\left[\frac{5}{(s+3)^2+25}\right]=2L^{-1}\left[\frac{s+3}{(s+3)^2+25}\right]-L^{-1}\left[\frac{1}{(s+3)^2+25}\right]$

$=2L^{-1}\left[\frac{s+3}{(s+3)^2+25}\right]-\frac{1}{5}L^{-1}\left[\frac{5}{(s+3)^2+25}\right]=2e^{-3t}\cos 5t-\frac{1}{5}e^{-3t}\sin 5t$

8. Determine the Laplace transform of the following functions.

(3 points) a. $\mathscr{L}\left\{e^{3t}\cos 4t + t^2 e^{2t}\right\}$ $= L[e^{3t}\cos 4t] + L[t^2 e^{2t}]$

$= \frac{s-3}{(s-3)^2+16} + \frac{2}{(s-2)^3}$

(3 points) b. $\mathscr{L}\left\{4t^3 + 6e^{4t} + 2\sin 5t\right\}$ $= L[4t^3] + L[6e^{4t}] + L[2\sin 5t]$

$= 4\frac{3!}{s^4} + 6\cdot\frac{1}{s-4} + 2\cdot\frac{5}{s^2+25}$

$= \frac{24}{s^4} + \frac{6}{s-4} + \frac{10}{s^2+25}$

7. A mass weighing 20 pounds stretches a spring 6 inches. The mass is initially released from rest from a point 6 inches below the equilibrium position.

(8 points) a. Find the equation of motion, $x(t)$. ($g = 32$ ft/s^2) $x(0) = \frac{1}{2}$ft $x'(0)=0$

$F = kx$

$20\text{lb} = k(\frac{1}{2}\text{ft})$

$40 = k$

$F = mg$

$20\text{lb} = m\,32$

$\frac{5}{8} = m$

$\frac{5}{8}\frac{d^2x}{dt^2} + 40x = 0$

$\frac{d^2x}{dt^2} + 64x = 0$

$m^2 + 64 = 0$

$m = \pm 8i$

$x(t) = c_1\cos 8t + c_2\sin 8t$

$x'(t) = -8c_1\sin 8t + 8c_2\cos 8t$

$\frac{1}{2} = c_1$ $0 = c_2$

$x(t) = \frac{1}{2}\cos 8t$

(4 points) b. Find the equation of motion in the form $x(t) = A\sin(\omega t + \phi)$.

$A = \frac{1}{2}$ $c_1 = 1/2$ $c_2 = 0$ $\tan\phi = \frac{1/2}{0}$ $\Rightarrow \phi = \pi/2$

$x(t) = \frac{1}{2}\sin(8t + \pi/2)$

(8 points) *8.* A mass weighing 10 pounds stretches a spring 2 feet. The mass is replaced with another mass that weighs 8 pounds. The entire system is submerged in a medium that offers a damping force that is numerically equal to the instantaneous velocity. Initially, the mass is displaced 4 inches above the equilibrium position and released from rest. Find the equation of motion, $x(t)$. What type of damped motion is this system?

$F = kx$

$10\,lb = k(2ft)$

$5\,\frac{lb}{ft} = k$

$F = mg$

$8\,lb = m(32\,ft/s^2)$

$\frac{1}{4}$ slugs $= m$

$\frac{1}{4}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 5x = 0$

$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 20x = 0$

$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(20)}}{2(1)}$

$= \frac{-4 \pm \sqrt{16-80}}{2}$

$= \frac{-4 \pm \sqrt{-64}}{2}$

$= \frac{-4 \pm 8i}{2} = -2 \pm 4i$

$x(t) = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t)$

$x(0) = -\frac{1}{3}ft \quad x'(0) = 0$

$-\frac{1}{3} = c_1$

$x'(t) = e^{-2t}(-4c_1 \sin 4t + 4c_2 \cos 4t)$

$\quad - 2e^{-2t}(c_1 \cos 4t + c_2 \sin 4t)$

$0 = 4c_2 - 2c_1$

$-\frac{2}{3} = 4c_2 \quad c_2 = -\frac{1}{6}$

$x(t) = e^{-2t}\left(-\frac{1}{3}\cos 4t - \frac{1}{6}\sin 4t\right)$

(10 points) *9.* Determine the general solution to the following system of differential equations.

$$\frac{dx_1}{dt} = -2x_1 - 7x_2$$

$$\frac{dx_2}{dt} = -x_1 + 4x_2$$

$\det\begin{bmatrix} -2-\lambda & -7 \\ -1 & 4-\lambda \end{bmatrix} = (-2-\lambda)(4-\lambda) - 7 = -8 - 2\lambda + \lambda^2 - 7 = \lambda^2 - 2\lambda - 15$

$\lambda^2 - 2\lambda - 15 = 0 \quad (\lambda-5)(\lambda+3) = 0 \quad \lambda = 5, -3$

$\lambda = 5$

$\begin{bmatrix} -7 & -7 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} -7x_1 - 7x_2 = 0 \\ -x_1 - x_2 = 0 \end{matrix} \rightarrow -x_1 - x_2 = 0 \quad \begin{matrix} x_2 = s \\ x_1 = -s \end{matrix} \quad \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = -3$

$\begin{bmatrix} 1 & -7 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 - 7x_2 = 0 \\ -x_1 + 7x_2 = 0 \end{matrix} \rightarrow x_1 - 7x_2 = 0 \quad \begin{matrix} x_2 = r \\ x_1 = 7r \end{matrix} \quad \begin{bmatrix} 7r \\ r \end{bmatrix} = r\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$Y = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{5t} + c_2\begin{bmatrix} 7 \\ 1 \end{bmatrix}e^{-3t}$

(10 points) *10.* Use Laplace transforms to solve the given initial value problem.

$$y'' - y = 12e^{2t}, \ y(0) = 0, \ y'(0) = 1$$

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] - \left[Y(s)\right] = \frac{12}{s-2}$$

$$s^2 Y(s) - 1 - Y(s) = \frac{12}{s-2}$$

$$Y(s)\left(s^2 - 1\right) = \frac{12}{s-2} + 1$$

$$Y(s) = \frac{12}{(s-2)(s+1)(s-1)} + \frac{1}{(s+1)(s-1)}$$

$$Y(s) = \frac{12 + s - 2}{(s-2)(s+1)(s-1)}$$

$$Y(s) = \frac{s+10}{(s-2)(s+1)(s-1)}$$

$$\frac{s+10}{(s-2)(s+1)(s-1)} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s-1}$$

$$s+10 = A(s+1)(s-1) + B(s-2)(s-1) + C(s-2)(s+1)$$

$s = 2$ $\quad 12 = 3A$ $\quad A = 4$

$s = -1$ $\quad 9 = -6B$ $\quad B = -\frac{3}{2}$

$s = 1$ $\quad 11 = -2C$ $\quad C = -\frac{11}{2}$

$$Y(s) = \frac{4}{s-2} + \frac{-3/2}{s+1} + \frac{-11/2}{s-1}$$

$$y(t) = 4e^{2t} + \frac{3}{2}e^{-t} - \frac{11}{2}e^{t}$$