MATH 290 – EXAM #3 Name: KEY
Summer Session 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(10 points) 12. Solve the following differential equation by using the superposition method.

$$y'' - 7y' + 12y = 3\cos 2x$$

$y_c = c_1 e^{3x} + c_2 e^{4x}$

$y_p = A\cos 2x + B\sin 2x$

$y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

$y_p'' - 7y_p' + 12y_p = (-4A\cos 2x - 4B\sin 2x) - 7(-2A\sin 2x + 2B\cos 2x) + 12(A\cos 2x + B\sin 2x) = 3\cos 2x$

$-4A\cos 2x - 4B\sin 2x + 14A\sin 2x - 14B\cos 2x + 12A\cos 2x + 12B\sin 2x = 3\cos 2x$

$(14A + 8B)\sin 2x + (8A - 14B)\cos 2x = 3\cos 2x$

$7(14A + 8B = 0)$

$4(8A - 14B = 3)$

$98A + 56B = 0$

$32A - 56B = 12$

$130A = 12$

$A = \frac{12}{130} = \frac{6}{65}$

$14\left(\frac{6}{65}\right) + 8B = 0$

$8B = -\frac{84}{65}$

$B = -\frac{21}{130}$

$y_p = \frac{6}{65}\cos 2x - \frac{21}{130}\sin 2x$

$y = c_1 e^{3x} + c_2 e^{4x} + \frac{6}{65}\cos 2x - \frac{21}{130}\sin 2x$

(10 points) 2. Solve the following differential equation by using annihilators.

$$y'' + 2y' - 8y = 4e^{2x} - 3x$$

$y_c = c_1 e^{-4x} + c_2 e^{2x}$

$(D^2 + 2D - 8)y = 4e^{2x} - 3x$

$(D+4)(D-2)y = 4e^{2x} - 3x$

$D^2(D+4)(D-2)^2 y = 0$

$y = c_1 + c_2 x + \cancel{c_3 e^{-4x}} + \cancel{c_4 e^{2x}} + c_5 x e^{2x}$

$y_p = A + Bx + Cxe^{2x}$

$y_p' = B + C(e^{2x} + 2xe^{2x})$

$y_p'' = C(2e^{2x} + 2e^{2x} + 4xe^{2x}) = C(4e^{2x} + 4xe^{2x})$

$C(4e^{2x} + 4xe^{2x}) + 2[B + C(e^{2x} + 2xe^{2x})] - 8[A + Bx + Cxe^{2x}] = 4e^{2x} - 3x$

$4Ce^{2x} + 4Cxe^{2x} + 2B + 2Ce^{2x} + 4Cxe^{2x} - 8A - 8Bx - 8Cxe^{2x} = 4e^{2x} - 3x$

$6Ce^{2x} - 8Bx + (2B - 8A) = 4e^{2x} - 3x$

$4 = 6C$ $\quad -8B = -3 \quad 2B - 8A = 0 \quad 8A = \frac{3}{4} \quad y_p = \frac{3}{32} + \frac{3}{8}x + \frac{2}{3}xe^{2x}$

$C = 2/3 \quad B = 3/8 \quad 2(\frac{3}{8}) - 8A = 0$

$\frac{3}{4} - 8A = 0 \quad A = \frac{3}{32} \quad y = c_1 e^{-4x} + c_2 e^{2x} + \frac{3}{32} + \frac{3}{8}x + \frac{2}{3}xe^{2x}$

3. Solve the following differential equations.

(4 points) a. $3x^2 y'' - xy' + 4y = 0$

$3m^2 + (-1-3)m + 4 = 0$

$3m^2 - 4m + 4 = 0$

$m = \frac{4 \pm \sqrt{16 - 4(3)(4)}}{2(3)}$

$m = \frac{4 \pm \sqrt{16-48}}{6}$

$m = \frac{4 \pm \sqrt{-32}}{6}$

$m = \frac{4 \pm 4i\sqrt{2}}{6}$

$m = \frac{2}{3} \pm \frac{2\sqrt{2}}{3}i$

$y = x^{\frac{2}{3}}\left[c_1\cos\left(\frac{2\sqrt{2}}{3}\ln x\right) + c_2\sin\left(\frac{2\sqrt{2}}{3}\ln x\right)\right]$

(4 points) b. $9x^2 y'' + 9xy' - y = 0$

$9m^2 + (9-9)m - 1 = 0$

$9m^2 - 1 = 0$

$m = \pm 1/3$

$y = c_1 x^{1/3} + c_2 x^{-1/3}$

(10 points) *4.* Solve the following equation by using variation of parameters.

$$y'' - 10y' + 25y = \frac{2e^{5x}}{4+x^2}$$

$y_c = c_1 e^{5x} + c_2 x e^{5x}$

$y_p = u_1 e^{5x} + u_2 x e^{5x}$

$u_1' e^{5x} + u_2' x e^{5x} = 0$

$5u_1' e^{5x} + u_2'(e^{5x} + 5xe^{5x}) = \frac{2e^{5x}}{4+x^2}$

$u_1' + u_2' x = 0$

$5u_1' + u_2'(1+5x) = \frac{2}{4+x^2}$

$\begin{vmatrix} 1 & x \\ 5 & 1+5x \end{vmatrix} = 1 + 5x - 5x = 1$

$u_1' = \begin{vmatrix} 0 & x \\ \frac{2}{4+x^2} & 1+5x \end{vmatrix} = -\frac{2x}{4+x^2}$

$u_1 = \int \left(\frac{-2x}{4+x^2}\right) dx = -\ln|4+x^2|$

$u_2' = \begin{vmatrix} 1 & 0 \\ 5 & \frac{2}{4+x^2} \end{vmatrix} = \frac{2}{4+x^2}$

$u_2 = \int \frac{2}{4+x^2}\,dx = 2\int \frac{1/4}{1+\left(\frac{x}{2}\right)^2} dx = \frac{1}{2}\int \frac{1}{1+\left(\frac{x}{2}\right)^2} dx$

$= \tan^{-1}\frac{x}{2}$

$y_p = -\ln|4+x^2|\, e^{5x} + x e^{5x} \tan^{-1}\frac{x}{2}$

$y = c_1 e^{5x} + c_2 x e^{5x} - e^{5x}\ln|4+x^2| + x e^{5x} \tan^{-1}\frac{x}{2}$

5. Give the annihilator operator for the following functions.

(2 points) *a.* $2x^2 e^{3x}\cos 2x$

$\left[D^2 - 2(3)D + (9+4)\right]^3$

$= (D^2 - 6D + 13)^3$

(2 points) *b.* $6e^{5x} - 5x^4 + 3xe^{2x}$

$D^5 (D-5)(D-2)^2$

(10 points) 6. Solve the following differential equation by using variation of parameters.

$$x^2y'' - 2xy' + 2y = x^4e^x \rightarrow y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x^2e^x$$

$m^2 + (-2-1)m + 2 = 0$

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$y_c = c_1 x + c_2 x^2$

$y_p = u_1 x + u_2 x^2$

$u_1' x + u_2' x^2 = 0$

$u_1' 1 + u_2' 2x = x^2 e^x$

$$\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

$$u_1' = \frac{\begin{vmatrix} 0 & x^2 \\ x^2e^x & 2x \end{vmatrix}}{x^2} = -\frac{x^4e^x}{x^2} = -x^2e^x$$

x^2	e^x
$2x$	e^x
2	e^x
0	e^x

$u_1 = -x^2e^x + 2xe^x - 2e^x$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & x^2e^x \end{vmatrix}}{x^2} = \frac{x^3e^x}{x^2} = xe^x$$

$u_2 = xe^x - e^x$

$y_p = (-x^2e^x + 2xe^x - 2e^x)x + (xe^x - e^x)x^2$

$= -x^3e^x + 2x^2e^x - 2xe^x + x^3e^x - x^2e^x$

$= x^2e^x - 2xe^x$

$y = c_1 x + c_2 x^2 + x^2e^x - 2xe^x$

7. Determine the inverse Laplace transform of the following function.

(3 points) a. $\mathscr{L}^{-1}\left\{\frac{5}{s^3} + \frac{3s}{s^2-4} - \frac{4}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{5 \cdot \frac{1}{2!} \cdot \frac{2!}{s^3}\right\} + \mathscr{L}^{-1}\left\{3 \cdot \frac{s}{s^2-4}\right\} - \mathscr{L}^{-1}\left\{4 \cdot \frac{1}{3} \cdot \frac{3}{s^2+9}\right\}$

$$= \frac{5}{2}t^2 + 3\cosh(2t) - \frac{4}{3}\sin(3t)$$

(3 points) b. $\mathscr{L}^{-1}\left\{\frac{2s+3}{s^2+6s}\right\}$

$$\frac{2s+3}{s(s+6)} = \frac{A}{s} + \frac{B}{s+6}$$

$2s+3 = A(s+6) + Bs$

$s=0$: $3 = 6A$, $A = 1/2$

$s=-6$: $-9 = -6B$, $B = 3/2$

$$\mathscr{L}^{-1}\left\{\frac{2s+3}{s^2+6s}\right\} = \mathscr{L}^{-1}\left\{\frac{1/2}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{3/2}{s+6}\right\}$$

$$= \frac{1}{2} + \frac{3}{2}e^{-6t}$$

8. Determine the Laplace transform of the following functions.

(3 points) a. $\mathscr{L}\{7\cosh 4t + 5 - 6\sin 8t\}$ $= \mathscr{L}\{7\cosh 4t\} + \mathscr{L}\{5\} - \mathscr{L}\{6\sin 8t\}$

$$= \frac{7s}{s^2-16} + \frac{5}{s} - \frac{48}{s^2+64}$$

(3 points) b. $\mathscr{L}\{4t^3 + 6e^{4t} + 2\sin 5t\}$ $= \mathscr{L}\{4t^3\} + \mathscr{L}\{6e^{4t}\} + \mathscr{L}\{2\sin 5t\}$

$$= 4 \cdot \frac{3!}{s^4} + \frac{6 \cdot}{s-4} + \frac{10}{s^2+25}$$

$$= \frac{24}{s^4} + \frac{6}{s-4} + \frac{10}{s^2+25}$$

(8 points) 9. Derive $\mathscr{L}\{t\} = \dfrac{1}{s^2}$.

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \lim_{b\to\infty}\int_0^b te^{-st}\,dt = \lim_{b\to\infty}\left(\frac{te^{-st}}{-s}\Big|_0^b + \frac{1}{s}\int_0^b e^{-st}\,dt\right)$$

$$= \lim_{b\to\infty}\left(\frac{te^{-st}}{-s} - \frac{1}{s^2}e^{-st}\right)\Big|_0^b$$

$$= \lim_{b\to\infty}\left(\frac{be^{-sb}}{-s} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2}\right)$$

$e^{-sb} \to 0$ for $s>0$

$e^{-sb} \to \infty$ for $s<0$.

$$\lim_{b\to\infty}\left(\frac{be^{-sb}}{-s} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2}\right) = \frac{1}{s^2}, \ s>0.$$

9. A mass weighing 12 pounds stretches a spring 2 feet. The mass is initially released from a point 1 foot below the equilibrium position with an upward velocity of 4 ft/s.

(8 points) *a.* Find the equation of motion, $x(t)$. ($g = 32$ ft/s^2)

$F = mg$

$12 = m(32)$

$\frac{12}{32} = m$

$\frac{3}{8}$ slugs $= m$

$F = kx$

$12 = k(2)$

$6 \frac{lb}{ft} = k$

$\frac{3}{8}\frac{d^2x}{dt^2} + 6x = 0$

$\frac{d^2x}{dt^2} + 16x = 0$

$m_1^2 + 16 = 0$

$m_1 = \pm 4i$

$x(t) = c_1 \cos 4t + c_2 \sin 4t$

$x(0) = 1$ ft, $x'(0) = -4$ ft/s

$x(0) = 1 = c_1$

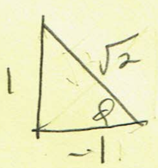
$x'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$

$x'(0) = -4 = 4c_2$

$c_2 = -1$

$x(t) = \cos 4t - \sin 4t$

(4 points) *b.* Find the equation of motion in the form $x(t) = A\sin(\omega t + \phi)$.



$\phi = 3\pi/4$

$A = \sqrt{2}$

$x(t) = \sqrt{2}\sin\left(4t + \frac{3\pi}{4}\right)$

(2 points) *c.* What is the period?

$T = \frac{2\pi}{4} = \frac{\pi}{2}$

(8 points) 10. A force of 2 pounds stretches a spring 1 foot. With one end held fixed, a mass weighing 8 pounds is attached to the other end. The system is submerged in a medium that offers a damping force that is numerically equal to $3/2$ times the instantaneous velocity. Initially, the mass is displaced 4 inches above the equilibrium position and released from rest. Find the equation of motion, $x(t)$. What type of damped motion is this system?

$F = kx$ $\quad F = mg$

$2 = k(1)$ $\quad 8 = m(32)$

$2\,\frac{lb}{ft} = k$ $\quad \frac{1}{4}\,slug = m$

$\frac{1}{4}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 2x = 0$

$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$

$m_1^2 + 6m_1 + 8 = 0$

$(m_1+2)(m_1+4) = 0 \quad m_1 = -2, -4$

$x = c_1 e^{-2t} + c_2 e^{-4t}$ $\quad x(0) = -\frac{1}{3}\,ft, \quad x'(0) = 0\,ft/s$

$x(0) = -\frac{1}{3} = c_1 + c_2$

$x'(t) = -2c_1 e^{-2t} - 4c_2 e^{-4t}$

$0 = -2c_1 - 4c_2$

$2(-1 = 3c_1 + 3c_2)$

$3(0 = -2c_1 - 4c_2)$

$-2 = 6c_1 + 6c_2$

$0 = -6c_1 - 12c_2$

$-2 = -6c_2 \quad c_2 = 1/3$

$c_1 = -2/3$

$x(t) = -\frac{2}{3}e^{-2t} + \frac{1}{3}e^{-4t}$

(8 points) 11. A mass weighing 6.4 pounds stretches a spring 3.2 feet. The entire system is placed in a medium that offers a damping force that is numerically equal to 1.2 times the instantaneous velocity. Find the equation of motion, $x(t)$, if the mass is initially released from rest from a point a point 6 inches below the equilibrium position. What type of motion is this?

$F = mg$ $\quad F = kx$

$6.4 = m(32)$ $\quad 6.4 = k(3.2)$

$0.2\,slug = m$ $\quad 2\,\frac{lb}{ft} = k$

$0.2\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 0$

$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 0$

$x = \frac{-6 \pm \sqrt{36-40}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i$

$x = e^{-3t}\left[c_1 \cos t + c_2 \sin t\right]$ $\quad x(0) = 1/2\,ft, \quad x'(0) = 0\,ft/s$

$x(0) = 1/2 = c_1$

$x'(t) = e^{-3t}\left[-\frac{1}{2}\sin t + c_2 \cos t\right] - 3e^{-3t}\left[\frac{1}{2}\cos t + c_2 \sin t\right]$

$x'(0):\quad 0 = c_2 - \frac{3}{2} \Rightarrow c_2 = 3/2$

$x(t) = e^{-3t}\left[\frac{1}{2}\cos t + \frac{3}{2}\sin t\right]$

(6 points extra credit) 12. Use Laplace transforms to solve the given initial value problem.

$$y'' - 4y = 6e^{3t}, \ y(0) = 0, \ y'(0) = 2$$

$$-y'(0) - sy(0) + s^2 Y(s) - 4Y(s) = \frac{6}{s-3}$$

$$-2 + (s^2-4)Y(s) = \frac{6}{s-3}$$

$$(s^2-4)Y(s) = \frac{6}{s-3} + 2$$

$$Y(s) = \frac{6}{(s-3)(s+2)(s-2)} + \frac{2}{(s+2)(s-2)}$$

$$Y(s) = \frac{6+2s-6}{(s-3)(s+2)(s-2)}$$

$$Y(s) = \frac{2s}{(s-3)(s+2)(s-2)}$$

$$\frac{2s}{(s-3)(s+2)(s-2)} = \frac{A}{s-3} + \frac{B}{s+2} + \frac{C}{s-2}$$

$$2s = A(s+2)(s-2) + B(s-3)(s-2) + C(s-3)(s+2)$$

$s = 3$ $\quad 6 = A(5)(1) \quad A = 6/5$

$s = -2$ $\quad -4 = B(-5)(-4) \quad B = -1/5$

$s = 2$ $\quad 4 = C(-1)(4) \quad C = -1$

$$Y(s) = \frac{6/5}{s+3} + \frac{-1/5}{s+2} + \frac{-1}{s-2}$$

$$y(t) = \frac{6}{5}e^{-3t} - \frac{1}{5}e^{-2t} - e^{2t}$$