Math 280 Exam #3

1. $y''-4y'+5y=e^{2x}\tan x$

7/ $m^2-4m+5=0$

$$m=\frac{-(-4)\pm\sqrt{16-4(1)(5)}}{2}$$

$$=\frac{4\pm\sqrt{16-20}}{2}$$

$$=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm 2i}{2}=2\pm i$$

$$y_c=e^{2x}(c_1\cos x+c_2\sin x)$$

$$y_p=e^{2x}(u_1\cos x+u_2\sin x)$$

$$u_1'(e^{2x}\cos x)+u_2'(e^{2x}\sin x)=0$$

$$u_1'(2e^{2x}\cos x-e^{2x}\sin x)+u_2'(2e^{2x}\sin x+e^{2x}\cos x)=e^{2x}\tan x$$

$$u_1'(\cos x)+u_2'(\sin x)=0$$

$$u_1'(2\cos x-\sin x)+u_2'(2\sin x+\cos x)=\tan x$$

$$W=\begin{vmatrix}\cos x & \sin x\\ 2\cos x-\sin x & 2\sin x+\cos x\end{vmatrix}$$

$$=\cos x(2\sin x+\cos x)-\sin x(2\cos x-\sin x)$$

$$=2\sin x\cos x+\cos^2 x-2\cos x\sin x+\sin^2 x=1$$

$$u_1'=\frac{\begin{vmatrix}0 & \sin x\\ \tan x & 2\sin x+\cos x\end{vmatrix}}{1}$$

$$u_1'=-\sin x\tan x=-\frac{\sin^2 x}{\cos x}=-\frac{(1-\cos^2 x)}{\cos x}$$

$$=-\sec x+\cos x$$

$$u_1=-\ln|\sec x+\tan x|+\sin x$$

$$u_2'=\frac{\begin{vmatrix}\cos x & 0\\ 2\cos x-\sin x & \tan x\end{vmatrix}}{1}$$

$$u_2'=\cos x\tan x=\sin x$$

$$u_2=-\cos x$$

$$y_p=e^{2x}\left((-\ln|\sec x+\tan x|+\sin x)\cos x+(-\cos x)\sin x\right)$$

$$=e^{2x}(-\cos x\ln|\sec x+\tan x|+\sin x\cos x-\sin x\cos x)$$

$$=e^{2x}(-\cos x\ln|\sec x+\tan x|)$$

$$y=e^{2x}(c_1\cos x+c_2\sin x)-e^{2x}\cos x\ln|\sec x+\tan x|$$

(2) $2x^2y'' + 7xy' + 3y = 0$

4/ $2m^2 + (7-2)m + 3 = 0$

$2m^2 + 5m + 3 = 0$

$(2m+3)(m+1) = 0$

$m = -3/2,\ m = -1$

$y = c_1 x^{-3/2} + c_2 x^{-1}$

(3) $2x^2y'' - 2xy' + 3y = 0$

4/ $2m^2 + (2-2)m + 3 = 0$

$2m^2 - 4m + 3 = 0$

$$m = \frac{4 \pm \sqrt{16 - 4(2)(3)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{16-24}}{4}$$

$$= \frac{4 \pm \sqrt{-8}}{4} = \frac{4 \pm 2i\sqrt{2}}{4}$$

$$= 1 \pm \frac{\sqrt{2}}{2} i$$

$$y = x\left(c_1 \cos \ln\left(\tfrac{\sqrt{2}}{2}x\right) + c_2 \sin \ln\left(\tfrac{\sqrt{2}}{2}x\right)\right)$$

$3x^2y'' + 7xy' + 2y = 0$

$3m^2 + (7-3)m + 2 = 0$

$3m^2 + 4m + 2 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 4(3)(2)}}{2(3)}$$

$$= \frac{-4 \pm \sqrt{16-24}}{6}$$

$$= \frac{-4 \pm \sqrt{-8}}{6} = \frac{-4 \pm 2i\sqrt{2}}{6} = -\frac{2}{3} \pm i\frac{\sqrt{2}}{3}$$

$$y = x^{-2/3}\left(c_1 \cos \ln\left(\tfrac{\sqrt{2}}{3}x\right) + c_2 \sin \ln\left(\tfrac{\sqrt{2}}{3}x\right)\right)$$

$3x^2y'' - 2xy' + 4y = 0$

$3m^2 + (-2-3)m + 4 = 0$

$3m^2 - 5m + 4 = 0$

$$m = \frac{5 \pm \sqrt{25 - 4(3)(4)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25-48}}{6}$$

$$= \frac{5 \pm \sqrt{-23}}{6} = \frac{5}{6} \pm i\frac{\sqrt{23}}{6}$$

$$y = x^{\frac{5}{6}}\left(c_1 \cos \ln\left(\tfrac{\sqrt{23}}{6}x\right) + c_2 \sin \ln\left(\tfrac{\sqrt{23}}{6}x\right)\right)$$

④ $9x^2y'' - 15xy' + 16y = 0$

$4/$ $9m^2 + (-15-9)m + 16 = 0$

$9m^2 - 24m + 16 = 0$

$(3m-4)^2 = 0$

$m = 4/3$

$y = c_1 x^{4/3} + c_2 x^{4/3} \ln x$

⑤ $x^2 y'' + 4xy' + 2y = 4\ln x$

$m^2 + (4-1)m + 2 = 0$

$\frac{4\ln x}{x^2}$

$7$ $m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = -2, -1$

$y_c = c_1 x^{-2} + c_2 x^{-1}$

$y_p = u_1 x^{-2} + u_2 x^{-1}$

$u_1' x^{-2} + u_2' x^{-1} = 0$

$u_1'(-2x^{-3}) + u_2'(-x^{-2}) = 4x^{-2}\ln x$

$$W = \begin{vmatrix} x^{-2} & x^{-1} \\ -2x^{-3} & -x^{-2} \end{vmatrix}$$

$= -x^{-4} + 2x^{-4} = x^{-4}$

$y = c_1 x^{-2} + c_2 x^{-1} + 2\ln x - 3$

$16x^2y'' - 8xy' + 9y = 0$

$16m^2 + (-8-16)m + 9 = 0$

$16m^2 - 24m + 9 = 0$

$(4m-3)^2 = 0$

$m = 3/4$

$y = c_1 x^{3/4} + c_2 x^{3/4} \ln x$

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ 4x^{-2}\ln x & -x^{-2} \end{vmatrix}}{x^{-4}}$$

$$u_1' = \frac{-4x^{-3}\ln x}{x^{-4}} = -4x\ln x$$

$u_1 = -4\int x\ln x\,dx$ $\qquad u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

$= -4\left[\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx\right]$

$= -4\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right] = -2x^2\ln x + x^2$

$$u_2' = \frac{\begin{vmatrix} x^{-2} & 0 \\ -2x^{-3} & 4x^{-2}\ln x \end{vmatrix}}{x^{-4}}$$

$$u_2' = \frac{4x^{-4}\ln x}{x^{-4}} = 4\ln x$$

$u_2 = 4\int \ln x\,dx = 4(x\ln x - x) = 4x\ln x - 4x$

$y_p = (-2x^2\ln x + x^2)x^{-2} + (4x\ln x - 4x)x^{-1}$

$= -2\ln x + 1 + 4\ln x - 4 = 2\ln x - 3$

⑥ $F = mg$ $\quad F = kx$

$32 = m\,32$ $\quad 32 = k(2)$

$m = 1$ slug $\quad k = 16$ lb/ft

$$1\frac{d^2x}{dt^2} + 16x = 0 \quad x(0) = -1 \quad x'(0) = -6$$

$n^2 + 16 = 0 \quad n = \pm 4i$

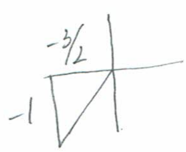
$x(t) = c_1 \cos 4t + c_2 \sin 4t$

$x'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$

$-1 = c_1 \quad -6 = 4c_2$

$c_2 = -3/2$

7/ $x(t) = -\cos 4t - \frac{3}{2}\sin 4t$




$\tan\phi' = \frac{1}{3/2} = 2/3$

$\phi' = 0.588$

2/ $\phi = \pi + 0.588 = 3.73$

$A = \sqrt{1 + (3/2)^2} = \frac{\sqrt{13}}{2}$

$x(t) = \frac{\sqrt{13}}{2}\sin(4t + 3.73)$

or

$x(t) = \frac{\sqrt{13}}{2}\sin(4t + 213.7°)$

$F = mg$ $\quad F = kx$

$32 = m\,32$ $\quad 32 = k(2)$

$m = 1$ slug $\quad k = 16$ lb/ft

$$1\frac{d^2x}{dt^2} + 16x = 0 \quad x(0) = -1 \quad x'(0) = -5$$

$n^2 + 16 = 0 \quad n = \pm 4i$

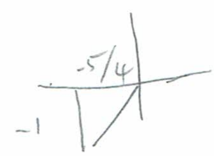
$x(t) = c_1 \cos 4t + c_2 \sin 4t$

$x'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$

$-1 = c_1 \quad -5 = 4c_2$

$c_2 = -5/4$

$x(t) = -\cos 4t - \frac{5}{4}\sin 4t$




$\tan\phi' = \frac{1}{5/4} = 4/5$

$\phi' = 0.6747$

$\phi = \pi + 0.6747 = 3.8163$

$A = \sqrt{1^2 + \left(\frac{5}{4}\right)^2} = \frac{\sqrt{41}}{4}$

$x(t) = \frac{\sqrt{41}}{4}\sin(4t + 3.8163)$

or

$x(t) = \frac{\sqrt{41}}{4}\sin(4t + 218.7°)$

⑦

$F = mg \qquad F = kx$

$4 = m(32) \qquad 4 = k(1)$

$m = \frac{1}{8}$ slug $\qquad k = 4$ lb/ft

$3.2 = m(32)$

$0.1$ slug $= m$

$0.1\frac{d^2x}{dt^2} + 0.4\frac{dx}{dt} + 4x = 0 \qquad x(0) = -1$

$x'(0) = 0$

$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 40x = 0$

7/ $\quad n^2 + 4n + 40 = 0$

$n = \frac{-4 \pm \sqrt{16 - 4(40)}}{2(1)}$

$= \frac{-4 \pm \sqrt{-144}}{2}$

$= \frac{-4 \pm 12i}{2} = -2 \pm 6i$

$x(t) = e^{-2t}(c_1 \cos 6t + c_2 \sin 6t)$

$x'(t) = -2e^{-2t}(c_1 \cos 6t + c_2 \sin 6t)$

$+ e^{-2t}(-6c_1 \sin 6t + 6c_2 \cos 6t)$

$-1 = c_1$

$0 = -2(-1) + 6c_2 \qquad c_2 = -1/3$

$x(t) = e^{-2t}(-\cos 6t - \frac{1}{3}\sin 6t)$

1/ underdamped motion

$F = mg \qquad F = kx$

$5 = m(32) \qquad 5 = k(1)$

$m = \frac{5}{32}$ slug $\qquad k = 5$ lb/ft

$6.4 = m(32)$

$m = 0.2$ slug

$0.2\frac{d^2x}{dt} + 1.6\frac{dx}{dt} + 5x = 0 \qquad x(0) = -1$

$\frac{d^2x}{dt} + 8\frac{dx}{dt} + 25x = 0 \qquad x'(0) = 0$

$n^2 + 8n + 25 = 0$

$n = \frac{-8 \pm \sqrt{64 - 4(25)}}{2(1)}$

$= \frac{-8 \pm \sqrt{-36}}{2}$

$= \frac{-8 \pm 6i}{2} = -4 \pm 3i$

$x(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)$

$x'(t) = -4e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)$

$+ e^{-4t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$

$-1 = c_1$

$0 = -4(-1) + 3c_2 \qquad c_2 = -4/3$

$x(t) = e^{-4t}(-\cos 3t - \frac{4}{3}\sin 3t)$

1/ underdamped motion.

(8) $F = mg$

$4 = m(32)$

$\frac{1}{8}$ slug $= m$

$x(0) = -1$

$x'(0) = 16$

$$\frac{1}{8}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = 0$$

$$\frac{d^2x}{dt} + 8\frac{dx}{dt} + 16x = 0$$

$n^2 + 8n + 16 = 0$

$(n+4)^2 = 0$

$n = -4$

7/ $x(t) = c_1 e^{-4t} + c_2 t e^{-4t}$

$x'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}$

$-1 = c_1$

$16 = 4 + c_2 \quad c_2 = 12$

$x(t) = -e^{-4t} + 12te^{-4t}$

1/ critically damped motion

$-e^{-4t} + 12te^{-4t} = 0$

2/ $-1 + 12t = 0$

$12t = 1$

$t = 1/12$ sec

$F = mg$

$4 = m(32)$

$m = \frac{1}{8}$ slug

$x(0) = 1$

$x'(0) = 12$

$$\frac{1}{8}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = 0$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$$

$n^2 + 8n + 16 = 0$

$(n+4)^2 = 0$

$n = -4$

$x(t) = c_1 e^{-4t} + c_2 t e^{-4t}$

$x'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 e^{-4t}$

$-1 = c_1$

$12 = 4 + c_2 \quad c_2 = 8$

$x(t) = -e^{-4t} + 12te^{-4t}$

critically damped motion

$-e^{-4t} + 12te^{-4t} = 0$

$-1 + 12t = 0$

$12t = 1$

$t = \frac{1}{12}$ sec

(9)

$$L\frac{di}{dt}+Ri+\frac{1}{C}q=E(t)$$

$$\frac{5}{4}\frac{di}{dt}+10i+40q=200$$

$$\frac{d^2q}{dt^2}+8\frac{dq}{dt}+32q=160$$

$$r^2+8r+32=0$$

$$r=\frac{-8\pm\sqrt{64-4(32)}}{2}$$

$$=\frac{-8\pm\sqrt{-64}}{2}=\frac{-8\pm 8i}{2}$$

$$r=-4\pm 4i$$

$$q_c(t)=e^{-4t}(c_1\cos 4t+c_2\sin 4t)$$

$$q_p(t)=A \quad q_p'=0 \quad q_p''=0$$

7/ $0+8(0)+32A=160$

$$A=5$$

$$q(t)=e^{-4t}(c_1\cos 4t+c_2\sin 4t)+5$$

$$0=c_1+5 \quad c_1=-5$$

$$q'(t)=i(t)=-4e^{-4t}(c_1\cos 4t+c_2\sin 4t)$$
$$+e^{-4t}(-4c_1\sin 4t+4c_2\cos 4t)$$

$$0=-4(-5)+4c_2$$

$$-20=4c_2 \quad c_2=-5$$

$$q(t)=e^{-4t}(-5\cos 4t-5\sin 4t)+5$$

2/ $i(t)=-4e^{-4t}(-5\cos 4t-5\sin 4t)$
$$+e^{-4t}(20\sin 4t-20\cos 4t)$$
$$=40\sin 4t$$

$$L\frac{di}{dt}+Ri+\frac{1}{C}q=E(t)$$

$$\frac{5}{2}\frac{di}{dt}+10i+20q=400$$

$$\frac{d^2q}{dt^2}+4i+8q=160$$

$$r^2+4r+8=0$$

$$r=\frac{-4\pm\sqrt{16-4(8)}}{2}$$

$$r=\frac{-4\pm\sqrt{-16}}{2}=\frac{-4\pm 4i}{2}$$

$$r=-2\pm 2i$$

$$q(t)=e^{-2t}(c_1\cos 2t+c_2\sin 2t)$$

$$q_p(t)=A \quad q_p'=0 \quad q_p''=0$$

$$0+4(0)+8A=160$$
$$8A=160$$
$$A=20$$

$$q(t)=e^{-2t}(c_1\cos 2t+c_2\sin 2t)+20$$

$$0=c_1+20 \quad c_1=-20$$

$$q'(t)=i(t)=-2e^{-2t}(c_1\cos 2t+c_2\sin 2t)$$
$$+e^{-2t}(-2\sin 2t+2c_2\cos 2t)$$

$$0=(-2)(-20)+2c_2$$

$$-40=2c_2 \quad c_2=-20$$

$$q(t)=e^{-2t}(-20\cos 2t-20\sin 2t)+20$$

$$i(t)=-2e^{-2t}(-20\cos 2t-20\sin 2t)$$
$$+e^{-2t}(40\sin 2t-40\cos 2t)$$
$$=80\sin 2t$$

(10)

3/ $\mathcal{L}\{5t^3+7\sinh(2t)+2e^{3t}\sin(4t)\}$

$$= 5\left(\frac{3!}{s^4}\right)+7\left(\frac{2}{s^2-4}\right)+2\left(\frac{4}{(s-3)^2+16}\right)$$

$$= \frac{30}{s^4}+\frac{14}{s^2-4}+\frac{8}{(s-3)^2+16}$$

$\mathcal{L}\{7t^4+4\cosh(2t)+5e^{2t}\sin(3t)\}$

$$= 7\left(\frac{4!}{s^5}\right)+4\left(\frac{s}{s^2-4}\right)+5\left(\frac{3}{(s-2)^2+9}\right)$$

$$= \frac{168}{s^5}+\frac{4s}{s^2-4}+\frac{15}{(s-2)^2+9}$$

(11)

$\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+4s+13}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+4s+4+13-4}\right\}$$

3| $= \mathcal{L}^{-1}\left\{\frac{2s+3}{(s+2)^2+9}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2s+3+1-1}{(s+2)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+9}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+9}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\}-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+9}\right\}$$

$$= 2e^{-2t}\cos 3t-\frac{1}{3}e^{-2t}\sin 3t$$

$\mathcal{L}^{-1}\left\{\frac{2s+7}{s^2+8s+65}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2s+7}{s^2+8s+16+65-16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s+7}{(s+4)^2+49}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s+7+1-1}{(s+4)^2+49}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+4)}{(s+4)^2+49}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+49}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s+4}{(s+4)^2+49}\right\}-\frac{1}{7}\mathcal{L}^{-1}\left\{\frac{7}{(s+4)^2+49}\right\}$$

$$= 2e^{-4t}\cos 7t-\frac{1}{7}e^{-4t}\sin 7t$$

(12) $y'' + y' - 2y = 10e^{-t} \quad y(0) = 0 \quad y'(0) = 1$

$$\left[-y'(0) - sy(0) + s^2 Y(s)\right] + \left[-Y(0) + sY(s)\right] - 2\left[Y(s)\right] = \frac{10}{s+1}$$

$$-1 + s^2 Y(s) + s Y(s) - 2Y(s) = \frac{10}{s+1}$$

$$Y(s)\left[s^2 + s - 2\right] = \frac{10}{s+1} + 1$$

$$Y(s)\ (s+2)(s-1) = \frac{10 + s + 1}{s+1}$$

$$Y(s) = \frac{s+11}{(s+1)(s+2)(s-1)}$$

$$\frac{s+11}{(s+1)(s+2)(s-1)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$s + 11 = A(s+2)(s-1) + B(s+1)(s-1) + C(s+1)(s+2)$$

$s = -1 \quad 10 = A(1)(-2) \quad A = -5$

$s = -2 \quad 9 = B(-1)(-3) \quad B = 3$

$s = 1 \quad 12 = C(2)(3) \quad C = 2$

$$Y(s) = \frac{-5}{s+1} + \frac{3}{s+2} + \frac{2}{s-1}$$

$$y(t) = -5e^{-t} + 3e^{-2t} + 2e^{t}$$

(13) $f(t) = 2 + 3\mathcal{U}(t-2) + t\,\mathcal{U}(t-4)$

$\mathcal{L}\{f(t)\}$

$= \mathcal{L}\{2\} + \mathcal{L}\{3\mathcal{U}(t-2)\} + \mathcal{L}\{t\,\mathcal{U}(t-4)\}$

3/ $= \mathcal{L}\{2\} + \mathcal{L}\{3\,\mathcal{U}(t-2)\} + \mathcal{L}\{(t-4+4)\mathcal{U}(t-4)\}$

$= \mathcal{L}\{2\} + \mathcal{L}\{3\,\mathcal{U}(t-2)\} + \mathcal{L}\{(t-4)\,\mathcal{U}(t-4)\} + \mathcal{L}\{4\,\mathcal{U}(t-4)\}$

$= \frac{2}{s} + \frac{3}{s}e^{-2s} + \frac{1}{s^2}e^{-4s} + \frac{4}{s}e^{-4s}$

(14) $\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-3s}\right\} = e^{4(t-3)}\mathcal{U}(t-3)$ $\quad\mathcal{L}^{-1}\left\{\frac{1}{s-3}e^{-4s}\right\} = e^{3(t-4)}\mathcal{U}(t-4)$

2/