MATH 290 – EXAM #3 **Name:** KEY
Winter Session 2019

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 103 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. Good luck!

(10 points) *1.* Solve the following differential equation by using the superposition method.

$$y'' - 16y = 2e^{4x}$$

$m^2 - 16 = 0$

$m = \pm 4$

$y_c = c_1 e^{4x} + c_2 e^{-4x}$

$y_p = Axe^{4x}$

$y_p' = Ae^{4x} + 4Axe^{4x}$

$y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x} = 8Ae^{4x} + 16Axe^{4x}$

$8Ae^{4x} + 16Axe^{4x} - 16Axe^{4x} = 2e^{4x}$

$8A = 2$

$A = \frac{1}{4}$

$y_p = \frac{1}{4}xe^{4x}$

$y = c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4}xe^{4x}$

(10 points) 2. Solve the following differential equation by using annihilators.

$$y'' + 6y' + 8y = 3e^{-2x} + 2x$$

$m^2 + 6m + 8 = 0$

$y_c = c_1 e^{-4x} + c_2 e^{-2x}$

Annihilator: $D^2(D+2)$

$D^2(D+2)^2(D+4)y = 0$

$y = c_1 + c_2 x + \cancel{c_3 e^{-2x}} + c_4 x e^{-2x} + c_5 e^{-4x}$

$y_p = A + Bx + Cxe^{-2x}$

$y_p' = B + Ce^{-2x} + C(-2xe^{-2x})$

$y_p'' = -2Ce^{-2x} + -2Ce^{-2x} + 4Cxe^{-2x}$

$= -4Ce^{-2x} + 4Cxe^{-2x}$

$-4Ce^{-2x} \cancel{+4Cxe^{-2x}} + 6B + 6Ce^{-2x} \cancel{-12Cxe^{-2x}}$

$+8A + 8Bx + \cancel{8Cxe^{-2x}} = 3e^{-2x} + 2x$

$(8A + 6B) + 8Bx + 2Ce^{-2x} = 3e^{-2x} + 2x$

$8A + 6B = 0 \quad 8B = 2 \quad 2C = 3$

$B = 1/4 \quad C = 3/2$

$8A = -3/2$

$A = -3/16$

$y_p = -3/16 + \frac{1}{4}x + \frac{3}{2}xe^{-2x}$

$y = c_1 e^{-4x} + c_2 e^{-2x} - \frac{3}{16} + \frac{1}{4}x + \frac{3}{2}xe^{-2x}$

(10 points) 3. Solve the following differential equation using any method.

$$y'' - y' - 2y = 10 \sin x$$

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$y_c = c_1 e^{2x} + c_2 e^{-x}$

Annihilator: $D^2 + 1$

$(D^2+1)(D-2)(D+1)y = 0$

$y = c_1 \cos x + c_2 \sin x + c_3 e^{2x} + c_4 e^{-x}$

$y_p = A\cos x + B \sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$-A\cos x - B\sin x + A\sin x - B\cos x - 2A\cos x - 2B\sin x$
$= 10\sin x$

$(-3B + A)\sin x + (-3A - B)\cos x = 10 \sin x$

$3(\ A - 3B = 10\ )$
$-3A - B = 0$

$3A - 9B = 30$
$-3A - B = 0$

$-10B = 30$
$B = -3$

$A + 9 = 10$
$A = 1$

$y_p = \cos x - 3\sin x$

$y = c_1 e^{2x} + c_2 e^{-x} + \cos x - 3\sin x$

4. Give the annihilator operator for the following functions.

(2 points) b. $x^2 + xe^{3x}$ $D^3(D-3)^2$

(2 points) c. $e^{2x}\cos 5x$ $D^2 - 2\cdot 2D + (2^2+5^2)$

$D^2 - 4D + 29$

(10 points) 5. Solve the following equation by using variation of parameters.

$$y'' + 6y' + 9y = \frac{2e^{-3x}}{x^2+1}$$

$y_c = c_1 e^{-3x} + c_2 x e^{-3x}$

$y_p = u_1 e^{-3x} + u_2 x e^{-3x}$

$u_1' e^{-3x} + u_2' x e^{-3x} = 0$

$-3u_1' e^{-3x} + u_2'(e^{-3x} - 3xe^{-3x}) = \frac{2e^{-3x}}{x^2+1}$

$3(\ u_1' + u_2' x = 0\ )$

$-3u_1' + u_2'(1-3x) = \frac{2}{x^2+1}$

$3u_1' + 3u_2' x = 0$

$-3u_1' + u_2'(1-3x) = \frac{2}{x^2+1}$

$u_2' = \frac{2}{x^2+1}$

$u_2 = 2\tan^{-1} x$

$u_1' = -\frac{2x}{x^2+1}$

$u_1 = -\ln|x^2+1|$

$y_p = -e^{-3x}\ln|x^2+1| + 2xe^{-3x}\tan^{-1} x$

$y = c_1 e^{-3x} + c_2 x e^{-3x} - e^{-3x}\ln|x^2+1| + 2xe^{-3x}\tan^{-1} x$

6. Solve the following differential equation.

(4 points) *a.* $4x^2y'' + 4xy' - y = 0$

$4m^2 - 1 = 0$

$m = \pm 1/2$

$y = C_1 x^{1/2} + C_2 x^{-1/2}$

(4 points) *b.* $x^2y'' + 7xy' + 9y = 0$

$m^2 + (7-1)m + 9 = 0$

$m^2 + 6m + 9 = 0$

$m = -3$ multiplicity 2

$y = c_1 x^{-3} + c_2 x^{-3} \ln x$

(4 points) *c.* $x^2y'' - 7xy' + 10y = 0$

$m^2 + (-7-1)m + 10 = 0$

$m^2 - 8m + 10 = 0$

$m = \dfrac{8 \pm \sqrt{64 - 4(10)}}{2(1)}$

$= \dfrac{8 \pm \sqrt{24}}{2}$

$= \dfrac{8 \pm 2\sqrt{6}}{2}$

$= 4 \pm \sqrt{6}$

$y = C_1 x^{4+\sqrt{6}} + C_2 x^{4-\sqrt{6}}$

7. A mass weighing 12 pounds stretches a spring 2 feet. The mass is initially released from a point 1 foot below the equilibrium position with an upward velocity of 4 ft/s.

(8 points) *a.* Find the equation of motion, $x(t)$. $(g = 32 \text{ ft/s}^2)$

$F = kx$

$12 = k \cdot 2$

$k = 6$

$12 = m \cdot 32$

$\frac{3}{8} = m$

$\frac{3}{8}\frac{d^2x}{dt^2} + 6x = 0$

$\frac{d^2x}{dt^2} + 16x = 0$

$r^2 + 16 = 0$

$r = \pm 4i$

$x(0) = 1 \text{ ft}$

$x'(0) = -4 \text{ ft/s}$

$x = c_1 \cos 4t + c_2 \sin 4t$

$x' = -4c_1 \sin 4t + 4c_2 \cos 4t$

$c_1 = 1$

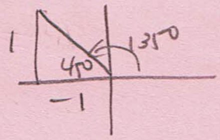
$-4 = 4c_2$

$c_2 = -1$

$x = \cos 4t - \sin 4t$

(4 points) *b.* Find the equation of motion in the form $x(t) = A\sin(\omega t + \phi)$.

$A = \sqrt{1 + (-1)^2} = \sqrt{2}$



$x(t) = \sqrt{2}\sin\left(4t + \frac{3\pi}{4}\right)$

(8 points) *8.* A mass weighing 2 pounds stretches a spring 1 foot. The mass is replaces with another mass that weighs 8 pounds. The systems is submerged in a medium that offers a damping force that is numerically equal to 3/2 times the instantaneous velocity. Initially, the mass is displaced 4 inches above the equilibrium position and released from rest. Find the equation of motion, $x(t)$. What type of damped motion is this system?

$F = kx$

$2 = k \cdot 1$

$k = 2$

$F = mg$

$8 = m \cdot 32$

$\frac{1}{4} = m$

$\frac{1}{4}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 2x = 0$

$x(0) = -\frac{1}{3}$

$x'(0) = 0$

$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$

$n^2 + 6n + 8 = 0$

$(n+4)(n+2) = 0$

$n = -4, -2$

$x = c_1 e^{-4t} + c_2 e^{-2t}$

$x' = -4c_1 e^{-4t} - 2c_2 e^{-2t}$

$2 \cdot (-\frac{1}{3} = c_1 + c_2)$

$0 = -4c_1 - 2c_2$

$-\frac{2}{3} = 2c_1 + 2c_2$

$0 = -4c_1 + 2c_2$

$-\frac{2}{3} = -2c_1$

$c_1 = \frac{1}{3}$

$-\frac{1}{3} = \frac{1}{3} + c_2$

$c_2 = -\frac{2}{3}$

$x = \frac{1}{3}e^{-4t} - \frac{2}{3}e^{-2t}$

Overdamped.

(5 points) *9.* A mass weighing 32 pounds stretches a spring 6 inches. The mass moves through a medium offering a damping force that is numerically equal to β times the instantaneous velocity. Determine the value of β for which the system will be critically damped.

$F = mg$

$32 = m(32)$

$1 \text{ slug} = m$

$F = kx$

$32 = k \cdot \frac{1}{2}$

$64 = k$

$$1\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 64x = 0$$

$$n = \frac{-\beta \pm \sqrt{\beta^2 - 4(1)(64)}}{2(1)}$$

$$\beta^2 - 256 = 0$$

$$\beta^2 = 256$$

$$\beta = 16$$

10. Determine the inverse Laplace transform of the following function.

(3 points) *a.* $\mathscr{L}^{-1}\left\{\frac{1}{s^3} - \frac{48}{s^5} + \frac{4}{s-2}\right\} = \frac{t^2}{2!} - \frac{48t^4}{4!} + 4e^{2t}$

$= \frac{1}{2}t^2 - 2t^4 + 4e^{2t}$

(3 points) *b.* $\mathscr{L}^{-1}\left\{\frac{s+2}{s^2+16}\right\} = L^{-1}\left\{\frac{s}{s^2+16}\right\} + L^{-1}\left\{\frac{2}{s^2+16}\right\}$

$= L^{-1}\left\{\frac{s}{s^2+16}\right\} + \frac{1}{2}L^{-1}\left\{\frac{4}{s^2+16}\right\}$

$= \cos 4t + \frac{1}{2}\sin 4t$

11. Determine the Laplace transform of the following functions.

(3 points) a. $\mathscr{L}\left\{e^{-4t}\sin 5t\right\}$ $= \dfrac{5}{(s+4)^2+25}, \quad s > -4$

(3 points) b. $\mathscr{L}\left\{t^4+8e^{7t}+2\cos 3t\right\}$ $= \dfrac{4!}{s^5} + \dfrac{8}{s-7} + \dfrac{2s}{s^2+9}, \quad s>7$

(10 points) 12. Use Laplace transforms to solve the given initial value problem.

$$y''+y'-2y=10e^{-t},\ y(0)=0,\ y'(0)=1$$

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+\left[sY(s)-y(0)\right]-2\left[Y(s)\right]=\frac{10}{s+1}$$

$$(s^2+s-2)Y(s)-1=\frac{10}{s+1}$$

$$(s+2)(s-1)Y(s)=\frac{10}{s+1}+1$$

$$(s+2)(s-1)Y(s)=\frac{10+s+1}{s+1}$$

$$Y(s)=\frac{s+11}{(s+1)(s+2)(s-1)}$$

$$Y(s)=-\frac{5}{s+1}+\frac{3}{s+2}+\frac{2}{s-1}$$

$$y(t)=-5e^{-t}+3e^{-2t}+2e^{t}$$

$$\frac{s+11}{(s+1)(s+2)(s-1)}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s-1}$$

$$s+11=A(s+2)(s-1)+B(s+1)(s-1)+C(s+1)(s+2)$$

$s=-1$ $\quad 10=A(1)(-2)$ $\quad A=-5$

$s=-2$ $\quad 9=B(-1)(-3)$ $\quad 3=B$

$s=1$ $\quad 12=C(2)(3)$ $\quad 2=C$