MATH 290 – EXAM #3 **Name:** KEY

Winter Session 2020

Directions: ***Please show all work for maximum credit.*** *No work = no credit.* Point values for each problem are given. There is a total of 102 points on this exam. This exam will be taken out of 100 points. Please show all work and clearly indicate your answers. Remember, this exam is to show what you know. You may not use any notes, the textbook, mobile phones, or any unauthorized sources for assistance during this exam. You may use a scientific calculator on this exam; however, you may not use a graphing calculator nor a multiview calculator. Clearly indicate the answer to each question. Any work on separate paper that you would like graded must be indicated on each corresponding problem on this exam. For application questions, use $g = 32 \text{ ft/s}^2$. Good luck!

(10 points) 1. Solve the following differential equation by using annihilators.

$$y'' - y' - 12y = 2xe^{4x}$$

$(D^2 - D - 12)y = 0$

$(D-4)(D+3)y = 0$

$y_c = c_1 e^{4x} + c_2 e^{-3x}$

$(D-4)(D+3)y = 2xe^{4x}$

$(D-4)^2(D-4)(D+3)y = (D-4)^2\, 2xe^{4x}$

$(D-4)^3(D+3)y = 0$

$y = c_1 e^{4x} + c_2 x e^{4x} + c_3 x^2 e^{4x} + c_4 e^{-3x}$

$y_p = Axe^{4x} + Bx^2e^{4x}$

$y_p' = Ae^{4x} + 4Axe^{4x} + B2xe^{4x} + B4x^2e^{4x} = Ae^{4x} + (4A+2B)xe^{4x} + 4Bx^2e^{4x}$

$y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x} + B2e^{4x} + 8xe^{4x} + B8xe^{4x} + B16x^2e^{4x}$

$= 8Ae^{4x} + 16Axe^{4x} + 2Be^{4x} + 16Bxe^{4x} + 16Bx^2e^{4x}$

$= (8A+2B)e^{4x} + (16A+16B)xe^{4x} + 16Bx^2e^{4x}$

$(8A+2B)e^{4x} + (16A+16B)xe^{4x} + 16Bx^2e^{4x}$

$- [Ae^{4x} + (4A+2B)xe^{4x} + 4Bx^2e^{4x}]$

$-12[Axe^{4x} + Bx^2e^{4x}] = 2xe^{4x}$

$(8A+2B-A)e^{4x} + (16A+16B-4A-2B-12A)xe^{4x}$

$+ (16B - 4B - 12B)x^2e^{4x} = 2xe^{4x}$

$7A + 2B = 0 \qquad 14B = 2$

$7A = -\frac{2}{7} \qquad B = \frac{1}{7}$

$A = -\frac{2}{49} \qquad y_p = -\frac{2}{49}xe^{4x} + \frac{1}{7}x^2e^{4x}$

$y = c_1 e^{4x} + c_2 e^{-3x} - \frac{2}{49}xe^{4x} + \frac{1}{7}x^2e^{4x}$

2. Solve the following differential equations.

(4 points) a. $3x^2y'' + 8xy' + 2y = 0$

$3m^2 + (8-3)m + 2 = 0$

$3m^2 + 5m + 2 = 0$

$(3m+2)(m+1) = 0$

$m = -2/3, -1$

$y = C_1 x^{-2/3} + C_2 x^{-1}$

(4 points) b. $4x^2y'' - y = 0$

$4m^2 + (0-4)m - 1 = 0$

$4m^2 - 4m - 1 = 0$

$m = \dfrac{4 \pm \sqrt{16 - 4(4)(-1)}}{2(4)} = \dfrac{4 \pm \sqrt{32}}{8} = \dfrac{4 \pm 4\sqrt{2}}{8}$

$= \dfrac{1}{2} \pm \dfrac{\sqrt{2}}{2}$

$y = C_1 x^{\left(\frac{1+\sqrt{2}}{2}\right)} + C_2 x^{\left(\frac{1-\sqrt{2}}{2}\right)}$

(4 points) c. $3x^2y'' - xy' + 4y = 0$

$3m^2 + (-1-3)m + 4 = 0$

$3m^2 - 4m + 4 = 0$

$m = \dfrac{4 \pm \sqrt{16 - 4(3)(4)}}{2(3)} = \dfrac{4 \pm \sqrt{16-48}}{6} = \dfrac{4 \pm \sqrt{-32}}{6} = \dfrac{4 \pm 4i\sqrt{2}}{6}$

$= \dfrac{2}{3} \pm \dfrac{2}{3}\sqrt{2}\, i$

$y = x^{2/3}\left(C_1 \cos\left(\frac{2\sqrt{2}}{3} \ln x\right) + C_2 \sin\left(\frac{2\sqrt{2}}{3} \ln x\right)\right)$

(4 points) d. $x^3y''' + xy' - y = 0$

$y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1)x^{m-2}$

$y''' = m(m-1)(m-2)x^{m-3}$

$x^3 m(m-1)(m-2)x^{m-3} + x m x^{m-1} - x^m = 0$

$m(m-1)(m-2)x^m + m x^m - x^m = 0$

$x^m\left[m(m-1)(m-2) + m - 1\right] = 0$

$x^m\left[(m-1)(m(m-2)+1)\right] = 0$

$x^m\left[(m-1)(m^2-2m+1)\right] = 0$

$x^m\left[(m-1)(m-1)(m-1)\right] = 0$

$m = 1$ mult. 3

$y = C_1 x + C_2 x \ln x + C_3 x (\ln x)^2$

(10 points) *3.* Solve the following equation by using variation of parameters.

$$y'' - 2y' + y = \frac{e^x}{1+x^2}$$

$y'' - 2y' + y = 0$

$(D^2 - 2D + 1)y = 0$

$y_c = C_1 e^x + C_2 x e^x$

$y_p = u_1 e^x + u_2 x e^x$

$u_1' e^x + u_2' x e^x = 0$

$u_1' e^x + u_2' (e^x + xe^x) = \frac{e^x}{1+x^2}$

$u_1' + u_2' x = 0$

$u_1' + u_2'(1+x) = \frac{1}{1+x^2}$

$W = \begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = 1 + x - x = 1$

$u_1' = \frac{\begin{vmatrix} 0 & x \\ \frac{1}{1+x^2} & 1+x \end{vmatrix}}{1} = -\frac{x}{1+x^2}$

$u_1 = -\int \frac{x}{1+x^2}\,dx = -\frac{1}{2}\ln|1+x^2|$

$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ 1 & \frac{1}{1+x^2} \end{vmatrix}}{1} = \frac{1}{1+x^2}$

$u_2 = \int \frac{1}{1+x^2}\,dx = \tan^{-1}x$

$y_p = -\frac{1}{2} e^x \ln|1+x^2| + xe^x \tan^{-1}x$

$y = C_1 e^x + C_2 x e^x - \frac{1}{2} e^x \ln(1+x^2) + xe^x \tan^{-1}x$

4. Give the annihilator operator for the following functions.

(2 points) *a.* $5x^2 e^{4x} \cos 3x$

$D^2 - 2\alpha D + (\alpha^2 + \beta^2)$

$(D^2 - 8D + 25)^3$

(2 points) *b.* $6e^{3x} + 5x^3 - 3x^2 e^{2x}$

$(D-3)D^4(D-2)^3$

(10 points) 5. Solve the following differential equation by using variation of parameters.

$$x^2 y'' - 4xy' + 6y = 2x^4 + x^2$$

$x^2y'' - 4xy' + 6y = 0$

$m^2 + (-4-1)m + 6 = 0$

$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$y_c = c_1 x^2 + c_2 x^3$

$y_p = u_1 x^2 + u_2 x^3$

$u_1' x^2 + u_2' x^3 = 0$

$u_1'(2x) + u_2'(3x^2) = 2x^2 + 1$

$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$

$= 3x^4 - 2x^4 = x^4$

$y'' - \frac{4y'}{x} + \frac{6}{x^2}y = 2x^2 + 1$

$u_1' = \dfrac{\begin{vmatrix} 0 & x^3 \\ 2x^2+1 & 3x^2 \end{vmatrix}}{x^4} = -\dfrac{2x^5 + x^3}{x^4} = -\left(2x + \frac{1}{x}\right)$

$u_1 = -\int \left(2x + \frac{1}{x}\right)dx = -x^2 - \ln|x|$

$u_2' = \dfrac{\begin{vmatrix} x^2 & 0 \\ 2x & 2x^2+1 \end{vmatrix}}{x^4} = \dfrac{2x^4 + x^2}{x^4} = 2 + \dfrac{1}{x^2}$

$u_2 = \int \left(2 + \frac{1}{x^2}\right)dx = 2x - \frac{1}{x}$

$y_p = (-x^2 - \ln|x|)x^2 + \left(2x - \frac{1}{x}\right)x^3 = -x^4 - x^2\ln|x| + 2x^4 - x^2$

$= x^4 - x^2\ln|x| - x^2$

$y = c_1 x^2 + c_2 x^3 + x^4 - x^2\ln|x| - x^2$

$y = c_3 x^2 + c_2 x^3 + x^4 - x^2 \ln|x|$

6. Determine the Laplace transform of the following functions.

(3 points) a. $\mathscr{L}\left\{5\cos 3t + 2e^{4t} - 7\right\}$

$= 5\left(\frac{s}{s^2+9}\right) + 2\left(\frac{1}{s-4}\right) - 7\left(\frac{1}{s}\right) = \frac{5s}{s^2+9} + \frac{2}{s-4} - \frac{7}{s}$

(3 points) b. $\mathscr{L}\left\{6\cosh 4t + 5t^3 - 8\sin 2t\right\}$

$= 6\left(\frac{s}{s^2-16}\right) + 5\left(\frac{3!}{s^4}\right) - 8\left(\frac{2}{s^2+4}\right)$

$= \frac{6s}{s^2-16} + \frac{30}{s^4} - \frac{16}{s^2+4}$

7. A mass weighing 24 pounds stretches a spring 4 inches. The mass is initially released from the equilibrium position with a downward velocity of 2 ft/s.

(8 points) *a.* Find the equation of motion, $x(t)$.

$W = mg$

$24 = m(32)$

$\frac{24}{32} = m$

$\frac{3}{4}\text{slug} = m$

$F = kx$

$24 = k\left(\frac{1}{3}\right)$

$72\,\frac{\text{lb}}{\text{ft}} = k$

$x(0) = 0\text{ ft}$

$x'(0) = 2\text{ ft/s}$

$m\frac{d^2x}{dt^2} + kx = 0$

$\frac{3}{4}\frac{d^2x}{dt^2} + 72x = 0$

$\frac{d^2x}{dt^2} + 96x = 0$

$n^2 + 96 = 0$

$n = \pm i\sqrt{96}$

$n = \pm 4i\sqrt{6}$

$x(t) = c_1 \cos 4\sqrt{6}t + c_2 \sin 4\sqrt{6}t$

$x'(t) = -4\sqrt{6}c_1 \sin 4\sqrt{6}t + 4\sqrt{6}c_2 \cos 4\sqrt{6}t$

$x(0) = 0$

$0 = c_1$

$x'(0) = 2$

$2 = 4\sqrt{6}c_2 \Rightarrow c_2 = \frac{2}{4\sqrt{6}} = \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}}{12}$

$x(t) = \frac{\sqrt{6}}{12}\sin 4\sqrt{6}t$

(4 points) *b.* Find the equation of motion in the form $x(t) = A\sin(\omega t + \phi)$.

$x(t) = \frac{\sqrt{6}}{12}\sin\left(4\sqrt{6}t\right)$

(2 points) *c.* What is the period?

$\frac{2\pi}{4\sqrt{6}}$

(8 points) *8.* A force of 2 points stretches a spring 1 foot. A mass weighing 3.2 pounds is attached to the spring, and the system is them immersed in a medium that offers a damping force that is numerically equal to 0.4 times the instantaneous velocity. Find the equation of motion, $x(t)$, if the mass is initially released from rest from a point 1 foot above the equilibrium position.

$F = kx$

$2\,lb = k(1\,ft)$

$2\,lb/ft = k$

$W = mg$

$3.2\,lb = m(32\,ft/s^2)$

$0.1\,slug = m$

$x(0) = -1\,ft$

$x'(0) = 0\,ft/s$

$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$

$0.1\frac{d^2x}{dt^2} + 0.4\frac{dx}{dt} + 2x = 0$

$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 20x = 0$

$n^2 + 4n + 20 = 0$

$n = \frac{-4 \pm \sqrt{16 - 4(20)}}{2(1)}$

$= \frac{-4 \pm \sqrt{-64}}{2} = \frac{-4 \pm 8i}{2} = -2 \pm 4i$

$x(t) = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t)$

$x'(t) = -2e^{-2t}(c_1 \cos 4t + c_2 \sin 4t)$

$+ e^{-2t}(-4c_1 \sin 4t + 4c_2 \cos 4t)$

$-1 = c_1$

$0 = -2(-1) + (4c_2)$

$-2 = 4c_2 \quad c_2 = -1/2$

$x(t) = e^{-2t}(-\cos 4t - \frac{1}{2}\sin 4t)$

underdamped.

(8 points) *9.* A mass weighing 32 pounds stretches a spring 6 inches. The mass moves through a medium offering a damping force that is numerically equal to 16 times the instantaneous velocity. Initially, the mass is displaced 4 inches below the equilibrium position and released with an upward velocity of 6 ft/s. Find the equation of motion, $x(t)$. What type of damped motion is this system?

$F = kx$

$32\,lb = k(\frac{1}{2}ft)$

$64\,lb/ft = k$

$W = mg$

$32\,lb = m(32\,ft/s^2)$

$1\,slug = m$

$x(0) = \frac{1}{3}ft$

$x'(0) = -6\,ft/s$

$\frac{d^2x}{dt} + 16\frac{dx}{dt} + 64x = 0$

$n^2 + 16n + 64 = 0$

$(n+8) = 0$

$n = -8$

$x(t) = c_1 e^{-8t} + c_2 te^{-8t}$

$x'(t) = -8c_1 e^{-8t} + c_2(e^{-8t} - 8te^{-8t})$

$\frac{1}{3} = c_1$

$-6 = -\frac{8}{3} + c_2$

$c_2 = -6 + \frac{8}{3} = \frac{-18+8}{3} = -\frac{10}{3}$

$x(t) = \frac{1}{3}e^{-8t} - \frac{10}{3}te^{-8t}$

critically damped.

(8 points) 10. A mass weighing 4 pounds is suspended from a spring whose constant is 3 lb/ft. The entire system is immersed in a fluid offering a damping force numerically equal to the instantaneous velocity. Beginning at $t=0$, an external force equal to $f(t)=e^{-t}$ is impressed on the system. Determine the equation of motion if the mass is initially released from rest at a point 2 feet below the equilibrium position.

$W = mg$

$4 = m(32)$

$\frac{1}{8}$ slug $= m$

$x(0) = 2\,ft$

$x'(0) = 0$

$\frac{1}{8}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x = e^{-t}$

$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 24x = 8e^{-t}$

$n^2 + 8n + 24 = 0$

$n = \frac{-8 \pm \sqrt{64 - 4(24)}}{2} = \frac{-8 \pm \sqrt{-32}}{2} = \frac{-8 \pm 4i\sqrt{2}}{2}$

$n = -4 \pm 2i\sqrt{2}$

$x(t) = e^{-4t}(c_1 \cos 2\sqrt{2}t + c_2 \sin 2\sqrt{2}t)$

$x_p = Ae^{-t}$

$x_p' = -Ae^{-t}$

$x_p'' = Ae^{-t}$

$Ae^{-t} - 8Ae^{-t} + 24Ae^{-t} = 8e^{-t}$

$17Ae^{-t} = 8e^{-t}$

$A = \frac{8}{17}$

$x(t) = e^{-4t}(c_1 \cos 2\sqrt{2}t + c_2 \sin 2\sqrt{2}t) + \frac{8}{17}e^{-t}$

$x'(t) = -4e^{-4t}(c_1 \cos 2\sqrt{2}t + c_2 \sin 2\sqrt{2}t)$
$+ e^{-4t}(-2\sqrt{2}c_1 \sin 2\sqrt{2}t + 2\sqrt{2}c_2 \cos 2\sqrt{2}t)$
$- \frac{8}{17}e^{-t}$

$x(0) = 2$

$2 = c_1 + \frac{8}{17}$ $\quad c_1 = \frac{26}{17}$

$x'(0) = 0$

$0 = -4\left(\frac{26}{17}\right) + 2\sqrt{2}c_2 - \frac{8}{17}$

$\frac{112}{17} = 2\sqrt{2}c_2$ $\quad c_2 = \frac{56}{17\sqrt{2}} = \frac{28\sqrt{2}}{17}$

$x(t) = e^{-4t}\left(\frac{26}{17}\cos 2\sqrt{2}t + \frac{28\sqrt{2}}{17}\sin 2\sqrt{2}t\right) + \frac{8}{17}e^{-t}$

(8 points) 11. Find the charge on the capacitor in an *LRC*-series circuit when the inductance is 1 henry, the resistance is 100 ohms, the capacitance is 0.0004 farad, and the electromotive force is $E(t) = 30$ V. The initial charge on the capacitor is $q(0) = 4$ C and the initial current is $i(0) = 2$ A.

$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

$1\frac{d^2q}{dt^2} + 100\frac{dq}{dt} + 2500q = 30$

$m^2 + 100m + 2500 = 0$

$(m+50)(m+50) = 0$

$m = -50$

$q(t) = c_1 e^{-50t} + c_2 te^{-50t}$

$q_p = A$

$q_p' = 0$

$q_p'' = 0$

$0 + 100(0) + 2500A = 30$

$A = \frac{30}{2500} = \frac{3}{250}$

$q(t) = c_1 e^{-50t} + c_2 te^{-50t} + \frac{3}{250}$

$4 = c_1 + \frac{3}{250}$ $\quad c_1 = 4 - \frac{3}{250} = \frac{997}{250}$

$q(t) = \frac{997}{250}e^{-50t} + c_2 te^{-50t} + \frac{3}{250}$

$q'(t) = -\frac{997}{5}e^{-50t} + c_2(e^{-50t} - 50te^{-50t})$

$2 = \frac{-997}{5} + c_2 \Rightarrow c_2\ \frac{1007}{5}$

$q(t) = \frac{997}{250}e^{-50t} + \frac{1007}{5}te^{-50t} + \frac{3}{250}$