MATH 290 – QUIZ #3 Name: KEY

Directions: This is a take home quiz. This quiz is due at the beginning of class on Monday, July 15, 2019. Please show all work for maximum credit. This quiz is worth 16 points. Good luck!

(5 points) 1. Given that $y_1(x) = e^{4x}$ is a solution to $y'' - 16y = 0$. Use reduction of order to find a second solution, $y_2(x)$.

$y_2 = uy_1$

$y_2 = ue^{4x}$

$y_2' = u'e^{4x} + 4ue^{4x}$

$y_2'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x}$

$(u''e^{4x} + 8u'e^{4x} + 16ue^{4x}) - 16(ue^{4x}) = 0$

$u''e^{4x} + 8u'e^{4x} = 0$

$e^{4x}(u'' + 8u') = 0$

$u'' + 8u' = 0$

let $w = u'$

$w' = u''$

$w' + 8w = 0$

$\mu = e^{\int 8dx} = e^{8x}$

$\frac{d}{dx}[e^{8x}w] = 0$

$e^{8x}w = C_1$

$w = C_1e^{-8x}$

$u' = C_1e^{-8x}$

$u = -\frac{1}{8}C_1e^{-8x} + C_2$

$y_2 = (-\frac{1}{8}C_1e^{-8x} + C_2)e^{4x}$

$= -\frac{1}{8}C_1e^{-4x} + C_2e^{4x}$

$y_2 = -\frac{1}{8}C_1e^{-4x}$ or $y_2 = C_3e^{4x}$

OR

$y_2 = y_1\int \frac{e^{\int P(x)dx}}{(y_1)^2}dx$

$y_2 = e^{4x}\int \frac{e^{\int 0dx}}{(e^{4x})^2}dx$

$= e^{4x}\int \frac{e^{C_1}}{e^{8x}}dx$

$= e^{4x}\int C_2e^{-8x}dx$

$= C_2e^{4x}\frac{e^{-8x}}{-8}$

$= Ce^{-4x}$

$y_2 = Ce^{-4x}$

(5 points) 2. Verify that $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$ form a fundamental set of solutions to the differential equation $y'' - 4y' + 4y = 0$ on the interval $(-\infty, \infty)$.

y_1: $y_1' = 2e^{2x}$, $y'' = 4e^{2x}$

$4e^{2x} - 4(2e^{2x}) + 4(e^{2x}) = 4e^{2x} - 8e^{2x} + 4e^{2x} = 0$ ✓

y_2: $y_2' = e^{2x} + 2xe^{2x}$, $y_2'' = 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x}$

$(4e^{2x} + 4xe^{2x}) - 4(e^{2x} + 2xe^{2x}) + 4(xe^{2x}) = 4e^{2x} + 4xe^{2x} - 4e^{2x} - 8xe^{2x} + 4xe^{2x} = 0$ ✓

$$\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{2x}(e^{2x} + 2xe^{2x}) - 2e^{2x}(xe^{2x})$$

$= e^{4x} + 2xe^{4x} - 2xe^{2x} = e^{4x} \neq 0$

$\therefore$ y_1 and y_2 are linearly independent

$\therefore$ y_1 and y_2 form a fundamental set of solutions.

(6 points) *3.* Two chemicals A and B are combined to form a chemical C. The rate of reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 50 grams of A and 30 grams of B, and for every 2 grams of A, 3 grams of B are used. It is observed that 10 grams of C are formed in 10 minutes. How much of C is formed in 20 minutes?

$$\frac{dx}{dt} \propto \left(50 - \frac{2}{5}x\right)\left(30 - \frac{3}{5}x\right)$$

$$\frac{dx}{dt} = k(125 - x)(50 - x)$$

$$\frac{dx}{(125-x)(50-x)} = k\,dt$$

$$\frac{1}{(125-x)(50-x)} = \frac{A}{125-x} + \frac{B}{50-x}$$

$$1 = A(50-x) + B(125-x)$$

$x = 125 \quad A = -\frac{1}{75}$

$x = 50 \quad B = \frac{1}{75}$

$$\int\left(\frac{-1/75}{125-x} + \frac{1/75}{50-x}\right)dx = \int k\,dt$$

$$\frac{1}{75}\ln|125-x| - \frac{1}{75}\ln|50-x| = kt + C_1$$

$$\frac{1}{75}\ln\left|\frac{125-x}{50-x}\right| = kt + C_1$$

$$\ln\left|\frac{125-x}{50-x}\right| = 75kt + 75C_1$$

$$\frac{125-x}{50-x} = Ce^{75kt}$$

$x(0) = 0g$

$$\frac{125}{50} = Ce^{75k(0)}$$

$$\frac{5}{2} = C$$

$$\frac{125-x}{50-x} = \frac{5}{2}e^{75kt}$$

$x(10) = 10g$

$$\frac{115}{40} = \frac{5}{2}e^{75k(10)}$$

$$\frac{115}{40} \cdot \frac{2}{5} = e^{750k}$$

$$\frac{23}{20} = e^{750k}$$

$$\frac{1}{750}\ln\frac{23}{20} = k$$

$$\frac{125-x}{50-x} = \frac{5}{2}e^{75\left(\frac{1}{750}\ln\frac{23}{20}\right)t}$$

$$\frac{125-x}{50-x} = \frac{5}{2}e^{\frac{1}{10}\left(\ln\frac{23}{20}\right)t}$$

$$\frac{125-x}{50-x} = \frac{5}{2}\left(\frac{23}{20}\right)^{\frac{1}{10}t}$$

$t = 20$

$$\frac{125-x}{50-x} = \frac{5}{2}\left(\frac{23}{20}\right)^{\frac{1}{10}(20)}$$

$$\frac{125-x}{50-x} = \frac{5}{2}\left(\frac{23}{20}\right)^2$$

$$125 - x = \frac{5}{2}\left(\frac{23}{20}\right)^2(50-x)$$

$$\frac{5}{2}\left(\frac{23}{20}\right)^2 x - x = \frac{5}{2}\left(\frac{23}{20}\right)^2 50 - 125$$

$$x = \frac{\frac{5}{2}\left(\frac{23}{20}\right)^2 50 - 125}{\frac{5}{2}\left(\frac{23}{20}\right)^2 - 1} \approx 17.48g.$$